Category Theory
Part III Michaelmas 2016-2017

Alexandre Daoud

May 27, 2017

Contents

1 Basic Definitions

Definition 1.1. A **category** $\mathcal{C}$ consists of

1. A collection of **objects**, denoted $\operatorname{ob}\mathcal{C}$.
2. A collection of **morphisms**, denoted $\operatorname{mor}\mathcal{C}$.
3. Two operations dom and codom from morphisms to objects such that, given two objects A and B and a morphism between them $f : A \to B$, we have $\operatorname{dom} f = A$ and $\operatorname{codom} f = B$.
4. An operation assigning to each object A an identity morphism $1_A : A \to A$.
5. A partial binary operation $(f, g) \mapsto gf$ such that gf is defined if and only if $\operatorname{dom} g = \operatorname{codom} f$ satisfying the following properties:
    - $f1_A = f$ and $1_B f = f$ for all $f : A \to B$.
    - $h(gf) = (hg)f$ whenever gf and hg are defined.

Definition 1.2. Let $\mathcal{C}$ be a category and $f : A \to B \in \operatorname{mor}\mathcal{C}$ a morphism. We say that f is an **isomorphism** if there exists a morphism $g : B \to A$ such that $fg = 1_B$ and $gf = 1_A$.

Remark. The definition of a category is not formalised within set theory. If $\operatorname{ob}\mathcal{C}$ and $\operatorname{mor}\mathcal{C}$ are sets (in some model of set theory) then we call $\mathcal{C}$ a **small** category. Furthermore, categories could be presented as a 1-sorted theory by identifying objects with their corresponding identity morphisms.

Example 1.3. The category **Sets** has all sets as objects and all functions between them as morphisms. Strictly speaking, the morphisms are pairs (f, B) where f is the set theoretic definition of the function and B is its codomain (since the set-theoretic definition of a function does not contain information about its codomain).

Example 1.4. The category **Grp** has all groups as objects and all group homomorphisms between them as morphisms. Similarly, **Ring** is the category whose objects are all rings and whose morphisms are all ring homomorphisms between them.

Example 1.5. The category **Top** has all topological spaces as objects and all continuous functions between them as morphisms.

Example 1.6. Given a category $\mathcal{C}$, we can construct a new category called the **opposite** category, denoted $\mathcal{C}^{\mathrm{op}}$ with the same objects and morphisms as $\mathcal{C}$ but with the domain and codomain maps interchanged and the order of composition reversed.

Example 1.7. A category with only one object is a monoid. For example, a group is a small category with one object where every morphism is an isomorphism. A category where every morphism is an isomorphism is called **groupoid**. For example, given any category $\mathcal{C}$, we can construct the groupoid $\operatorname{Iso}\mathcal{C}$ with the same objects as $\mathcal{C}$ but only isomorphisms of $\mathcal{C}$ as morphisms.

Example 1.8. The fundamental groupoid $\Pi(X)$ of a space X has points of X as objects and morphisms $x \to y$ are homotopy classes relative to $[0,1]$ of paths $a : [0,1] \to X$ with $a(0) = x$ and $a(1) = y$.

Example 1.9. A **discrete** category is one whose only morphisms are the identities.

Example 1.10. A category $\mathcal{C}$ in which there is at most one morphism from A to B for each pair of objects (A, B) is called a **preorder**. If, in addition, every isomorphhism is an identity then we have a **partial order**.

Example 1.11. The category **Rel** has the same objects as **Sets** but morphisms $A \to B$ are relations (subsets of $A \times B$. The composite $A \xrightarrow{R} B \xrightarrow{S} C$ is given by

$$S \cdot R = \{\, (a, c) \mid (\exists\, b \in B)((a, b) \in R \wedge (b, c) \in S) \,\}$$

Moreover, the category **Part** has partial functiions as morphisms. In other words, relations satisfying the uniqueness condition of functions but not necessarily the existence condition.

Example 1.12. Given a field K, the category $\mathbf{Mat_K}$ has natural numbers as objects and the morphisms $n \to p$ are $p \times n$ matrices with entries in K and composition given by matrix multiplication.

Definition 1.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of

1. A mapping $A \mapsto FA$ from $\operatorname{ob}\mathcal{C}$ to $\operatorname{ob}\mathcal{D}$

2. A mapping $f \mapsto Ff$ from $\operatorname{mor}\mathcal{C}$ to $\operatorname{mor}\mathcal{D}$

such that F commutes with the domain and codomain maps, $F(1_A) = 1_{FA}$ for all $A \in \operatorname{ob}\mathcal{A}$, and $F(gf) = (Fg)(Ff)$ whenever fg is defined.

Remark. We shall write **Cat** to refer to the category of small categories and functors between them.

Example 1.14. The **forgetful functor** $U : \mathbf{Grp} \to \mathbf{Set}$ sends a group to its underlying set and a homomorphism to itself.

Example 1.15. The functor $f : \mathbf{Rng} \to \mathbf{Grp}$ sending a ring to its group of units.

Example 1.16. Given a set A,let PA be its power set. We can make P into a functor $\mathbf{Set} \to \mathbf{Set}$. Indeed, given $f : A \to B$, we can define $Pf(A') = \{ f(a) \mid a \in A' \} \subseteq B$. We also have a functor $P^* : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ defined by $P^*A = PA$ and $P^*f(B') = \{ a \in A \mid f(a) \in B' \}$.

Definition 1.17. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We define a **contravariant** functor from $\mathcal{C}$ to $\mathcal{D}$ to be a functor $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Example 1.18. Given a field K, the dual of a vector space V over K is the vector space V^* of linear maps $V \to K$. Given $f : V \to W$ the dual function $f^* : V^* \to W^*$ is given by $\theta \mapsto \theta \circ f$. This makes $(\cdot)^*$ into a functor from $\mathbf{Mod}_K^{\mathrm{op}}$ to $\mathbf{Mod}_K$. Similarly, we have a functor $\mathbf{Rel}^{\mathrm{op}}$ to $\mathbf{Rel}$ sending a set A to itself and a relation $R \subseteq A \times B$ to $\mathbf{R}^{\mathrm{op}} = \{ (b, a) \in B \times A \mid (a, b) \in R \}$

Example 1.19. The construction $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is a functor $\mathbf{Cat} \to \mathbf{Cat}$.

Example 1.20. A functor between monoids is a monoid homomoprhism, in particular, a functor between groups is a group homomorphism (so **Grp** is a subcategory of **Cat**).

Example 1.21. A functor between posets is a map that preserves order.

Example 1.22. Given a group G, a functor $G \to \mathbf{Set}$ is a permutation representation of G. Similarly, a functor $G \to \mathbf{Mod}_K$ is a K-linear representation of G.

Example 1.23. The assignment $(X, x) \to \Pi_1(X, x)$ is a functor $\mathbf{Top}_* \to \mathbf{Grp}$ (infact, $\mathbf{Htpy}_* \to \mathbf{Grp}$). Similarly, the (singular) homology groups are functors $H_\bullet : \mathbf{Htpy} \to \mathbf{Grp}$.

Definition 1.24. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ two functors. A **natural transformation** $\alpha : F \to G$ assigns to each $A \in \operatorname{ob}\mathcal{C}$ a morphism $\alpha_A : FA \to GA$ in $\mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

for all $f : A \to B \in \operatorname{mor}\mathcal{C}$.

Example 1.25. Consider the power set functor $P : \mathbf{Set} \to \mathbf{Set}$. There is a natural transformation $\eta : 1_{\mathbf{Set}} \to P$ given by $\eta_A(a) = \{ a \}$. This is indeed natural since, given two sets $A, B \in \operatorname{ob}\mathbf{Set}$ and a function $f : A \to B \in \operatorname{mor}\mathbf{Set}$ we have

Example 1.26. Let G and H be groups and $f, g : G \rightrightarrows H$ homomorphisms. A natural transformation $\alpha : f \to g$ picks out a particular element $b \in H$ such that $hf(x) = g(x)h$ for all $x \in G$. In other words, f and g are conjugate homomorphisms.

Definition 1.27. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We write $[\mathcal{C}, \mathcal{D}]$ for the category of all functors and natural transformations between them.

Lemma 1.28. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ functors between them and $\alpha : F \to G$ a natural transformation. Then α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ if and only if each α_A is an isomorphism in $\mathcal{D}$.*

Proof. The forward direction is trivial. Assume that each α_A has an inverse $\beta_A : GA \to FA$. We need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow{\scriptstyle \beta_A} & & \downarrow{\scriptstyle \beta_B} \\ FA & \xrightarrow{Ff} & FB \end{array}$$

is commutative for all f. We have that

$$(Ff)\beta_A = \beta_B \alpha_B (Ff)\beta_A = \beta_B (Gf)\alpha_A \beta_A = \beta_B (Gf)$$

as required. □

Definition 1.29. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. We say that

1. F is **faithful** if, given $f, g \in \operatorname{mor} \mathcal{C}$, the equations $\operatorname{dom} f = \operatorname{dom} g, \operatorname{codom} f = \operatorname{codom} g$ and $Ff = Fg$ imply that $f = g$.
2. F is **full** if, given any $g : FA \to FB$ in $\mathcal{D}$, there exists $f : A \to B$ in $\mathcal{C}$ with $Ff = g$.
3. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is **full** if the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

Example 1.30. Grp is a full subcategory of the category of monoids. However, **Mon** is a non-full subcategory of the category **Sgrp** of semi-groups.

Definition 1.31. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We define an **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ to be a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$. We denote this equivalence by $C \simeq D$.

Furthermore, we say that a property P of categories is **categorical** if whenever $\mathcal{C}$ has property P and $\mathcal{C} \simeq \mathcal{D}$ then $\mathcal{D}$ has property P.

Example 1.32. Being a groupoid is categorical while being a group is not.

Example 1.33. Given an object B of a category $\mathcal{C}$, we define the **slice category**, denoted $\mathcal{C}/B$ to have morphisms $f : A \to B$ in $\mathcal{C}$ as objects and morphisms defined as follows: given two objects $f : A \to B$ and $g : C \to B$, $h : A \to C$ is a morphism of $\mathcal{C}/B$ provided that the diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & B \end{array}$$

commutes.

The category $\mathbf{Set}/B$ is equivalent to the category $\mathbf{Set}^B$ of B-indexed families of sets. We have a functor $F : \mathbf{Set}/B \to \mathbf{Set}^B$ which sends an object $f : A \to B$ to $(f^{-1}(b) | b \in B)$ and its inverse is given by $G : \mathbf{Set}^B \to \mathbf{Set}/B$ which sends $(A_b | b \in B)$ to $\coprod_{b \in B} A_b = \bigcup_{b \in B} A_b \times \{ b \}$.

Example 1.34. The category **Part** of sets and partial functions is equivalent to the category $\mathbf{Set}_*$ of pointed sets. Indeed, define a functor $F : \mathbf{Set}_* \to \mathbf{Part}$ that sends the object (A, a) to $A \backslash \{ 0 \}$ and the morphism $f : (A, a) \to (B, b)$ to the partial function sending $x \in A$ to $f(x)$ if $f(x) \neq b$ and undefined otherwise. Conversely, we have the functor $G : \mathbf{Part} \to \mathbf{Set}_*$ which sends the object A to $(A \cup \{ a \}, a)$ and $f : A \to B$ to Gf defined by $Gf(x) = f(x)$ if $x \in A$ and $f(x)$ is defined and B otherwise.

Example 1.35. The category $\mathbf{fdMod_K}$ of finite-dimensional vector spaces over a field K is equivalent to $\mathbf{fdMod_K}^{\mathrm{op}}$ via the dualisation functor $(\cdot)^*$ and the natural isomorphism $\alpha : 1_{\mathbf{fdMod_K}} \to (\cdot)^{**}$.

Example 1.36. We also have an equivalence $\mathbf{fMod_K} \cong \mathbf{Mat}_K$. Indeed, consider the function $F : \mathbf{Mat_K} \to \mathbf{fdMod_K}$ which sends $n \in \operatorname{ob} \mathbf{Mat_K}$ to K^n and a matrix $A \in \operatorname{mor} \mathbf{Mat_K}$ to the linear map represented by A with respect to the standard bases of the spaces. The functor in the opposite direction is defined similarly.

Lemma 1.37. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then F is part of an equivalence $\mathcal{C} \cong \mathcal{D}$ if and only if F is full, faithful and essentially surjective (for all $B \in \operatorname{ob} \mathcal{D}$, there exists $A \in \operatorname{ob} \mathcal{C}$ such that $FA \cong B$).*

Proof. First assume that $\mathcal{C} \cong \mathcal{D}$. Let $G : \mathcal{D} \to \mathcal{C}$, $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$ be the second functor and the natural isomorphisms that induce this equivalence. Then F is clearly essentially surjective as we have

$$FGB \cong 1_{\mathcal{D}} B \cong B$$

We next show that F is faithful. To this end, we must show that given $f, g : A \to B \in \operatorname{mor} \mathcal{C}$ satisfying $Ff = Fg$ we have $f = g$. By the naturality of α we have the commutative diagram

$$\begin{array}{ccc} GFA & \xrightarrow{GFf = GFg} & GFB \\ {\scriptstyle \alpha_A}\uparrow & & {\scriptstyle \alpha_B}\uparrow \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

Similarly, we get a commutative diagram for g. We thus have

$$\begin{aligned} f &= \alpha_B^{-1}(GFf)\alpha_A \\ &= \alpha_B^{-1}(GFg)\alpha_A \\ &= g \end{aligned}$$

and so F is faithful. We next show that F is full. To this end, we must show that given $g : FA \to FB \in \operatorname{mor} \mathcal{D}$, there exists $f : A \to B \in \operatorname{mor} \mathcal{C}$ such that $Ff = g$. We claim that $f = \alpha_A(Gg)\alpha_B^{-1}$ is the desired morphism. Observe that $Gg = \alpha_A^{-1}(f)\alpha_B$. Furthermore, by the naturality of α we have the commutative diagram

$$\begin{array}{ccc} GFA & \xrightarrow{GFf} & GFB \\ {\scriptstyle \alpha_A}\uparrow & & {\scriptstyle \alpha_B}\uparrow \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

and so $GFf = \alpha_A^{-1}(f)\alpha_B$. We hence have $Gg = GFf$. By the same argument as above, G is faithful and so $Ff = g$ as desired.

Conversely, suppose that F is essentially surjective and fully faithful. We need to show that $\mathcal{C} \cong \mathcal{D}$. To this end, we must exhibit a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$.

For all $D \in \operatorname{ob} \mathcal{D}$, there exists a $C \in \operatorname{ob} \mathcal{C}$ such that $FC \cong D$ by the essential surjectivity of F. We define a functor $G : \mathcal{D} \to \mathcal{C}$ that acts on objects of $\mathcal{D}$ by setting $GD = C$. We shall also use the essential surjectivity of F to choose an isomorphism $\beta_D : FGD \to D$ for all $D \in \operatorname{ob} \mathcal{D}$. We must now define G on $\operatorname{mor} \mathcal{D}$. Given $g : X \to Y \in \operatorname{mor} \mathcal{G}$, we define Gg to be the unique morphism in $\operatorname{mor} \mathcal{C}$ whose image under F is the composition

$$FGX \xrightarrow{\beta_X} X \xrightarrow{g} Y \xrightarrow{\beta_Y^{-1}} FGY$$

This definition of Gg ensures that β is a natural isomorphism. We must check that this is functorial. Given any other morphism $h : Y \to Z$ such that the composition gh is defined, the faithfulness of F implies that $G(g) \circ G(h) = G(gh)$.

Finally, we shall define $\alpha : 1_{\mathcal{C}} \to GF$. Given $A \in \mathcal{C}$, define $\alpha_A : A \to GFA$ to be the unique morphism in $\mathcal{C}$ satisfying $F\alpha_A = \beta_{FA}^{-1}$. This is clearly an isomorphism since the unique morphism in $\mathcal{C}$ that is mapped to β_{FA} is a two-sided inverse for it. It remains to show that α is a natural transformation. To this end, consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ GFA & \xrightarrow{GFf} & GFB \end{array}$$

We need to show that this diagram is commutative. Applying the functor F to the diagram yields

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \beta_{FA}^{-1} = F\alpha_A}\downarrow & & \downarrow{\scriptstyle F\alpha_B = \beta_{FB}^{-1}} \\ FGFA & \xrightarrow{FGFf} & FGFB \end{array}$$

which commutes by the naturality of β. In particular, we have

$$\begin{aligned} F(\alpha_B \circ f) &= F\alpha_B \circ Ff \\ &= \beta_{FB}^{-1} \circ Ff \\ &= \beta_{FA}^{-1} \circ FGFf \\ &= F(F\alpha_A \circ GFf) \end{aligned}$$

whence the faithfulness of F implies that $\alpha_B \circ f = F\alpha_A \circ GFf$ and so the original diagram is commutative and α is natural. □

Definition 1.38. Let $\mathcal{C}$ be a category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$. We say that $\mathcal{C}'$ is a **skeleton** of $\mathcal{C}$ if $\mathcal{C} \cong \mathcal{C}'$ and no two distinct objects of $\mathcal{C}'$ are isomorphic. Furthermore, we say that $\mathcal{C}$ is **skeletal** if it is a skeleton of itself.

Example 1.39. Given a field K, $\mathbf{Mat_K}$ is skeletal. Indeed, the full subcategory of standard vector-spaces K^n is a skeleton of $\mathbf{fdMod_K} \cong \mathbf{Mat_K}$.

Definition 1.40. Let $\mathcal{C}$ be a category and $f : A \to B \in \operatorname{mor} \mathcal{C}$ a morphism.

1. We say that f is a **monomorphism** if, given $g, h : C \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. In this case, we say that f is **monic** and we write $f : A \rightarrowtail B$.

2. We say that f is an **epimorphism** if, given $k, l : B \rightrightarrows D$, the equation $kf = lf$ implies $k = l$. In this case, we say that f is **epic** and we write $f : A \twoheadrightarrow B$.

3. We say that $\mathcal{C}$ is **balanced** if every epic and monic $f \in \operatorname{mor} \mathcal{C}$ is an isomorphism.

Example 1.41. Let $f \in \operatorname{mor} \mathbf{Set}$. Then f is monic if and only if it is injective. Indeed, first suppose that f is injective. Let $g, h : C \to A$ be such that $fg = fh$. Choose $x \in C$. Then we havea $f(g(x)) = f(h(x))$. But f is injective and so $g(x) = h(x)$. But x was arbitrary so $g = h$ and f is monic. Conversely, suppose that f is monic. Let $C = \{ z \}$ be a one-element set. Let $x, y \in A$ such that $f(x) = f(y)$. Let $g, h : C \rightrightarrows A$ be functions such that $g(z) = x, h(z) = y$. Then since f is monic, we have that $f(g(z)) = f(h(z))$ implies that $g(z) = h(z)$ and so $x = y$ whence f is injective.

Furthermore, f is epic if and only if it is surjective. Indeed, first suppose that f is surjective. We need to show that given $k, l : B \rightrightarrows D$ such that $kf = lf$ then $k = l$. Since f is surjective, for all $b \in B$, there exists $a \in A$ such that $f(a) = b$. Then $k(f(a)) = l(f(a))$ whence $k(b) = l(b)$ for all $b \in B$ and so $k = l$. Conversely, suppose that f is epic. We prove by contradiction so assume that there exists a $b \in B$ such that there exists no $a \in A$ with $f(a) = b$. Define $D = \{ 0, 1 \}$ and the functions $k, l : B \rightrightarrows D$ satisfying $k(x) = 1$ for all $x \in B$ and $l(x) = 1$ for all $x \in B$ except when $x = b$. We have that $k(f(a)) = l(f(a))$ for all $ainA$. Since f is epic, it follows that $k = l$ which is clearly a contradiction. Hence f must be surjective.

We thus see that **Set** is balanced.

Example 1.42. In **Grp**, we also have that $f \in \operatorname{mor} \mathbf{Grp}$ is injective (resp. surjective) if and only if f is monic (resp. epic). The proof for monicity is similar to that of sets using $\mathbb{Z}$ instead of a one-point set. Hence **Grp** is balanced.

Example 1.43. In **Rng**, we have that ring homomorphisms are monic if and only if they are surjective. However, we do not have that an epic ring homomorphism is necessarily surjective. Indeed, consider the inclusion $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Let R be any ring and $k, l : \mathbb{Q} \to R$ ring homomorphisms. Suppose that $kf = lf$. Then for any integer m we clearly have $k(m) = l(m)$. Now let $a/b \in \mathbb{Q}$. Then

$$k\left(\frac{a}{b}\right) = \frac{k(a)}{k(b)} = \frac{l(a)}{l(b)} = l\left(\frac{a}{b}\right)$$

and so $k = l$ whence f is a non-surjective epimorphism. Hence **Rng** is not balanced.

Example 1.44. In **Top** we also have $f \in \operatorname{mor} \mathbf{Top}$ is monic (resp. epic) if and only if f is injective (resp. surjective). However, **Top** is not balanced as a bijective continuous map does not necessarily have a continuous inverse and thus monic epimorphisms are not necessarily isomorphisms.

2 The Yoneda Lemma

Definition 2.1. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is **locally small** if for all $A, B \in \operatorname{ob} \mathcal{C}$, the morphisms are parametrised by a set $\mathcal{C}(A, B)$.

Remark. If $\mathcal{C}$ is a locally small category then we have the **hom-functor**

$$\begin{aligned} \mathcal{C}(A, \cdot) : \mathcal{C} &\to \mathbf{Set} \\ B &\mapsto \mathcal{C}(A, B) \end{aligned}$$

which assigns to every $B \in \operatorname{ob}\mathcal{C}$ the **hom-set** $\mathcal{C}(A,B)$. Similarly the assignment $A \mapsto \mathcal{C}(A,B)$ becomes a functor $\mathcal{C}^{\text{op}} \to \mathbf{Set}$.

Theorem 2.2 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a locally small category, $A \in \operatorname{ob}\mathcal{C}$ an object and $F : \mathcal{C} \to \mathbf{Set}$ a functor.*

1. *There exists a bijection between natural transformations $\mathcal{C}(A,\cdot) \to F$ and elements of FA.*
2. *Such a bijection is natural in both A and F.*

Proof.

Part 1: Given a natural transformation $\alpha : \mathcal{C}(A,\cdot) \to F$, define $\Phi(\alpha) = \alpha_A(1_A) \in FA$. Conversely, given $x \in FA$, define a natural transformation $\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$. We must first check that $\Psi(x)$ is indeed natural. To this end, let $g : B \to C$ be a morphism in $\mathcal{C}$. We must check that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}(A,B) & \xrightarrow{\mathcal{C}(A,g)} & \mathcal{C}(A,C) \\ \downarrow{\scriptstyle \Psi(x)_B} & & \downarrow{\scriptstyle \Psi(x)_C} \\ FB & \xrightarrow{Fg} & FC \end{array}$$

so that $\Psi(x)$ is a well-defined natural transformation. Fix $f \in \mathcal{C}(A,B)$. We have that

$$\begin{aligned} \Psi(x)_C(\mathcal{C}(A,g)(f)) &= \Psi(x)_C(g \circ f) \\ &= F(g \circ f)(x) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} Fg(\Psi(x)_B(f)) &= Fg(Ff(x)) \\ &= F(g \circ f)(x) \end{aligned}$$

and so the diagram is indeed commutative. We now claim that Φ and Ψ are mutually inverse. Given $x \in FA$ we have

$$\Phi(\Psi(x)) = \Psi(x)(1_A) = F(1_A)(x) = 1_{FA}(x) = x$$

Conversely, given a natural transformation $\alpha : \mathcal{C}(A,\cdot) \to F$ and $f : A \to B$, the naturality of α implies that

$$\begin{aligned} \Psi(\Phi(\alpha))_B(f) &= Ff(\Phi(\alpha)) \\ &= Ff(\alpha_A(1_A)) \\ &= \alpha_B(\mathcal{C}(A,f)(1_A)) \\ &= \alpha_B(f) \end{aligned}$$

and so $\Psi(\Phi(\alpha)) = \alpha$.

Part 2: We shall prove this part of the theorem in the case that $\mathcal{C}$ is small (so that $[\mathcal{C}, \mathbf{Set}]$ is locally small).

We have two functors

$$\begin{aligned} \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] &\to \mathbf{Set} \\ (A,F) &\mapsto FA \end{aligned}$$

$$\begin{aligned} \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] &\to \mathbf{Set} \\ (A,F) &\mapsto [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A,\cdot), F) \end{aligned}$$

where $[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, \cdot), F)$ is understood to be the set of all natural transformations between $\mathcal{C}(A, \cdot)$ and F. The statement of this part of the theorem asserts that Φ and Ψ are natural isomorphisms between these two functors. We first show naturality in A. Suppose $f : A' \to A$ is a morphism in $\mathcal{C}$. We need to show that the following diagram is commutative

$$\begin{array}{ccc} [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A', \cdot), F) & \xrightarrow{\Theta} & [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, \cdot), F) \\ \downarrow{\scriptstyle \Phi_{A'}} & & \downarrow{\scriptstyle \Phi_A} \\ FA' & \xrightarrow{\quad Ff \quad} & FA \end{array}$$

where $\Theta(\alpha) = \alpha \circ \mathcal{C}(f, \cdot)$ for a natural transformation $\alpha \in [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A', \cdot), F)$. Let $\alpha : \mathcal{C}(A', \cdot) \to F$ be a natural transformation. On one hand we have, by naturality of α,

$$Ff(\Phi_{A'}(\alpha)) = Ff(\alpha_{A'}(1_{A'})) = \alpha_A(f)$$

On the other hand, we have

$$\Phi_A(\Theta(\alpha)) = \Phi_A(\alpha \circ \mathcal{C}(f, \cdot)) = (\alpha \circ \mathcal{C}(f, \cdot)_A(1_A) = \alpha_A(f)$$

and hence Φ is natural in A. We now show that Φ is natural in F. To this end, let $\eta : F \to G$ be a natural transformation of functions $F, G : \mathcal{C} \rightrightarrows \mathbf{Set}$. We need to show that the following diagram is commutative

$$\begin{array}{ccc} [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, \cdot), F) & \xrightarrow{\eta \circ -} & [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, \cdot), G) \\ \downarrow{\scriptstyle \Phi_F} & & \downarrow{\scriptstyle \Phi_G} \\ FA & \xrightarrow{\quad \eta_A \quad} & GA \end{array}$$

Fix $\alpha : \mathcal{C}(A, \cdot) \to F$. On one hand, we have

$$\phi_G(\eta \circ \alpha) = (\eta \circ \alpha)_A(1_A) = \eta_A(\alpha_A(1_A))$$

On the other hand, we have

$$\eta_A(\Phi_F(\alpha)) = \eta_A(\alpha_A(1_A))$$

and hence Φ is natural in F. □

Corollary 2.3. *Let $\mathcal{C}$ be a locally small category. Then the mapping*

$$\begin{aligned} Y : \mathcal{C}^{\mathrm{op}} &\to [\mathcal{C}, \mathbf{Set}] \\ A &\mapsto \mathcal{C}(A, \cdot) \\ (f : B \to A) &\mapsto \mathcal{C}(f, \cdot) \end{aligned}$$

is a full and faithful functor referred to as the ***Yoneda embedding****. In particular, every locally small category is equivalent to a full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.*

Proof. We claim that Y is given on morphisms by the inverse of the bijective mapping

$$\Phi : [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, \cdot), \mathcal{C}(B, \cdot)) \to \mathcal{C}(B, A)$$

from Theorem 2.2. If we prove this claim, we automatically obtain a full and faithful functor.

To this end, let $\alpha : \mathcal{C}(A, \cdot) \to \mathcal{C}(B, \cdot)$ be a natural transformation. Then

$$Y(\Phi(\alpha)) = Y(\alpha_A(1_A)) = \mathcal{C}(\alpha_A(1_A), \cdot)$$

is a natural transformation. Given $g : A \to C$ we have

$$Y(\Phi(\alpha))_C(g) = \mathcal{C}(\alpha_A(1_A), C)(g) = g \circ \alpha_A(1_A)$$

Since α is a natural transformation, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}(A,A) & \xrightarrow{\mathcal{C}(A,g)} & \mathcal{C}(A,C) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_C} \\ \mathcal{C}(B,A) & \xrightarrow{\mathcal{C}(B,g)} & \mathcal{C}(B,C) \end{array}$$

and so

$$g \circ \alpha_A(1_A) = \alpha_C(g \circ 1_A) = \alpha_C(1_A)$$

We thus have

$$Y(\Phi(\alpha))_C(g) = \alpha_C(1_A)$$

whence Y is a left-inverse for Φ. Since Φ is bijective, Y must be a two-sided inverse as desired. The statement about equivalences then follows immediately upon realising F is essentially surjective onto its image. $\square$

Definition 2.4. Let $\mathcal{C}$ be a category. We say that a functor $F : \mathcal{C} \to \mathbf{Set}$ is **representable** if it is naturally isomorphic to $\mathcal{C}(A, \cdot)$ for some $A \in \text{ob}\,\mathcal{C}$. We define a **representation** of a representable functor $F : \mathcal{C} \to \mathbf{Set}$ to be a pair (A, x) where $A \in \text{ob}\,\mathcal{C}$, $x \in FA$ and $\Psi(x) : \mathcal{C}(A, \cdot) \to F$ is a natural isomorphism. Furthermore, x is referred to as a **universal element** of F.

Proposition 2.5. *Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathbf{Set}$ a functor. If (A, x) and (B, y) are representations of F then there is a unique isomorphism $f : A \to B$ such that $Ff(x) = y$*

Proof. Consider the composition

$$\mathcal{C}(B, \cdot) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi^{-1}(x)} \mathcal{C}(A, \cdot)$$

This is a natural isomorphism between the functors $\mathcal{C}(B, \cdot)$ and $\mathcal{C}(A, \cdot)$ and so the Yoneda Lemma implies that this natural isomorphism is given by a unique morphism $f : A \to B$. We thus have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(B,\cdot) & \xrightarrow{\mathcal{C}(f,\cdot)} & \mathcal{C}(A,\cdot) \\ & {\scriptstyle \Psi(y)} \searrow \quad \swarrow {\scriptstyle \Psi(x)} & \\ & F & \end{array}$$

This yields a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(B,B) & \xrightarrow{\mathcal{C}(f,B)} & \mathcal{C}(A,B) \\ & {\scriptstyle \Psi(y)_B} \searrow \quad \swarrow {\scriptstyle \Psi(x)_B} & \\ & FB & \end{array}$$

We now chase $1_B \in \mathcal{C}(B, B)$ around the diagram in two ways:

$$\begin{aligned} \Psi(x)_B(\mathcal{C}(f,B)(1_B)) &= \Psi(x)_B(f) = Ff(x) \\ \Psi(y)_B(1_B) &= F(1_B)(y) = 1_{FB}(y) = y \end{aligned}$$

and so $Ff(x) = y$ as required. $\square$

Example 2.6. The forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$ is represented by $(\mathbb{Z}, 1)$. Indeed, consider the natural transformation

$$\Psi(1) : \mathcal{C}(\mathbb{Z}, \cdot) \to F$$

given by $\Psi(1)_B(\mathbb{Z} \xrightarrow{\phi} B) = (F\phi)(1)$. We claim that $\Psi(1)$ is a natural isomorphism. To this end, fix a group $G \in \operatorname{ob}\mathbf{Grp}$ and let $\phi, \varphi : \mathbb{Z} \rightrightarrows G$. We have that

$$\Psi(1)_B(\phi) = \Psi(1)_B(\varphi) \iff (F\phi)(1) = (F\varphi)(1) \implies \phi(1) = \varphi(1) \implies \phi = \varphi$$

since $1 \in \mathbb{Z}$ is a generator. Hence $\Psi(1)_B$ is injective. Furthermore, given $g \in FG$, there is a unique homomorphism $\phi : \mathbb{Z} \to G$ mapping $1 \mapsto g$ and so $\Psi(1)_B$ is surjective. Hence $\Psi(1)$ is a natural isomorphism and the forgetful functor F is represented by $(\mathbb{Z}, 1)$

Example 2.7. The forgetful functor $F : \mathbf{Rng} \to \mathbf{Set}$ is represented by $(\mathbb{Z}[X], X)$. This is shown in the exact same way as the previous example.

Example 2.8. Let τ be any singleton topological space and x its unique element. Then the forgetful functor $F : \mathbf{Top} \to \mathbf{Set}$ is represented by (τ, x). Indeed, we have a natural transformation

$$\Psi(x) : \mathcal{C}(\tau, \cdot) \to F$$

given by $\Psi(x)_{\tau'}(\tau \xrightarrow{f} \tau') = (Ff)(x)$. We claim that $\Psi(x)$ is a natural isomorphism. To this end, fix a topological space $\tau \in \operatorname{ob}\mathbf{Top}$ and $f, g : \tau \to \tau'$ continuous maps. Then

$$\Psi(x)_{\tau'}(f) = \Psi(x)_{\tau'}(g) \implies (Ff)(x) = (Fg)(x) \implies f(x) = g(x) \implies f = g$$

whence $\Psi(x)_{\tau'}$ is injective. Furthermore, fix an element $y \in \tau'$. Then there is a unique continuous map $f : \tau \to \tau'$ taking x to y and so $\Psi(x)_{\tau'}$ is surjective. Thus $\Psi(x)$ is a natural isomorphism and the forgetful functor F is represented by (τ, x).

Example 2.9. The contravariant power set functor

$$P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$$

is represented by $(\{0,1\}, \{1\})$. Indeed, we have a natural transformation

$$\Psi(\{1\}) : \mathcal{C}(\{0,1\}, \cdot) \to F$$

given by $\Psi(\{1\})_B(\{0,1\} \xrightarrow{f} B) = (Ff)(\{1\})$. We claim that $\Psi(\{1\})$ is a natural isomorphism. To this end, fix a set B and $f, g : \{0,1\} \to B$ morphisms in $\mathbf{Set}^{\mathrm{op}}$. Then

$$\Psi(\{1\})_B(f) = \Psi(\{1\})_B(g) \iff (Ff)(\{1\}) = (Fg)(\{1\}) \implies \{f(1)\} = \{g(1)\}$$

Now, f and g are given by set theoretic functions $f, g : B \rightrightarrows 0, 1$ in $\mathbf{Set}$. $\{f(1)\} = \{g(1)\}$ can then be interpreted in $\mathbf{Set}$ to mean that the collection of elements in B that map to 1 under f and g are the same. This completely determines the images of the rest of the elements of B (they have to be 0) and so f and g define the same function in $\mathbf{Set}$ and thus they define the same morphism in $\mathbf{Set}^{\mathrm{op}}$. In other words, $\Psi(\{1\})_B$ is an injective mapping. Now, given an element B' of the power set of B, the characteristic function of B' in $\mathbf{Set}$ represents a morphism in $\mathbf{Set}^{\mathrm{op}}$ which maps to B' under $\Psi(\{1\})_B$ and so $\Psi(\{1\})_B$ is also surjective. It then follows that $\Psi(\{1\})$ is a natural isomorphism whence P^* is represented by $(\{0,1\}, \{1\})$.

Example 2.10. Let K be a field. Then the functor G given by the composition of the dual vector space functor $(\cdot)^* : \mathbf{Mod_K}^{\mathrm{op}} \to \mathbf{Mod_K}$ and the forgetful functor $\mathbf{Mod_K} \to \mathbf{Set}$ is represented by $(K, 1_K)$. Indeed, we have a natural transformation

$$\Psi(1_K) : \mathcal{C}(K, \cdot) \to G$$

given by $\Psi(1_K)_V(K \xrightarrow{\phi} V) = (G\phi)(1_K)$. We claim that $\Psi(1_K)$ is a natural isomorphism. To this end, let $V \in \mathrm{ob}\,\mathbf{Mod_K}$ be a vector space and $\phi, \psi : K \rightrightarrows V$ linear maps. Then

$$\Psi(1_K)_V(\phi) = \Psi(1_K)_V(\psi) \iff (G\phi)(1_K) = (G\psi)(1_K) \iff 1_K \circ \phi = 1_K \circ \psi \iff \phi = \psi$$

Furthermore, if $\psi_{GV} \in GV$ is a function $\psi : V \to K$ arising from a linear map ψ_{V^*} in V^* then ψ itself represents a morphism $K \xrightarrow{\psi_{\mathbf{Mod_K}^{\mathrm{op}}}} V$ in $\mathbf{Mod}_K^{\mathrm{op}}$. Such a $\psi_{\mathbf{Mod_K}^{\mathrm{op}}}$ maps to ψ_{GV} under $\Psi(1_K)_V$ and so $\Psi(1_K)_V$ is surjective whence $\Psi(1_K)$ is a natural isomorphism and G is representable by $(K, 1_K)$.

Definition 2.11. Let $\mathcal{C}$ be a locally small category and $f, g : A \rightrightarrows B$ morphisms of $\mathcal{C}$. Consider the functor

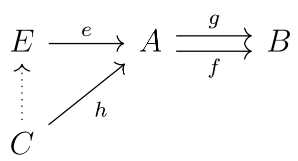
$$\mathcal{E}_{f,g} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$
$$C \mapsto \{\, (C \xrightarrow{h} A) \in \mathrm{mor}\,\mathcal{C} \mid fh = gh \,\}$$

which is a covariant subfunctor of $\mathcal{C}^{\mathrm{op}}(A, \cdot)$ (as a functor $\mathcal{C} \to \mathbf{Set}$, it is a contravariant subfunctor of $\mathcal{C}(\cdot, A)$). A representation (if it exists) of $\mathcal{E}_{f,g}$ is called an **equaliser** of f and g. Dually, we define the notion of a **coequaliser** to be a representation of the functor

$$\mathcal{F} : \mathcal{C} \to \mathbf{Set}$$
$$C \mapsto \{\, h : B \to C \mid hf = hg \,\}$$

We say that $\mathcal{C}$ **has equalisers** (resp. **has coequalisers**) if every pair of morphisms f and g in $\mathcal{C}$ has an equaliser.

Proposition 2.12. *Let $\mathcal{C}$ be a locally small category, $f, g : A \rightrightarrows B$ morphisms in $\mathcal{C}$ and $e : E \to A$ a morphism such that $fe = ge$. Then (E, e) is an equaliser for f and g if and only if h factors uniquely through e for all $(C \xrightarrow{h} A) \in \mathrm{mor}\,\mathcal{C}$ such that $fh = gh$. In other words, we have a commutative diagram*



Proof. Let (E, e) be a representation of $\mathcal{E}_{f,g}$. Then e is an element of $\mathcal{E}_{f,g}E$. In other words, $e : E \to A$ is a morphism in $\mathcal{C}$ satisfying $fe = ge$. By definition of a representation, we have a natural isomorphism

$$\Psi(e) : \mathcal{C}^{\mathrm{op}}(E, \cdot) \to \mathcal{E}_{f,g}$$

Hence for all morphisms $(C \xrightarrow{h} A) \in \mathcal{E}_{f,g}C$, there exists a unique morphism $(E \xrightarrow{k'} C) \in \mathrm{mor}\,\mathcal{C}^{\mathrm{op}}$ such that $\Psi(e)_C(k') = h$. Such a k' corresponds to a morphism $(C \xrightarrow{k} E) \in \mathrm{mor}\,\mathcal{C}$ with $ek = h$ as desired.

The backwards implication can be proven by reversing the above argumentation to construct a natural isomorphism $\Psi(e)$. $\square$

Example 2.13. Let $f, g : A \rightrightarrows B \in \operatorname{mor} \mathbf{Set}$. An equalizer for f and g is given by the set

$$E = \{\, a \in A \mid f(a) = g(a) \,\}$$

together with the inclusion $i : E \hookrightarrow A$. Indeed, suppose that $h : C \to A \in \operatorname{mor} \mathbf{Set}$ is another function satisfying $fh = gh$. We need to find a function $j : C \to E$ satsifying $ij = h$. We may simply take

$$\begin{aligned} j : C &\to E \\ c &\mapsto i^{-1}(h(c)) \end{aligned}$$

which is well-defined since the range of h is necessarily a subset of E.

Hence forth, all definitions or results labelled with a $\star$ are to be understood as dualisable. For example, whenever a monomorphism appears, replace it with an epimorphism. Whenever an equaliser appears, replace it when a coequaliser. Note that dualising often involves reversing the direction of arrows in diagrams.

Proposition 2.14 ($\star$)**.** *Let $\mathcal{C}$ be a category and $f, g : A \rightrightarrows B$ morphisms in $\mathcal{C}$. If $e : E \to A$ is an equaliser for f and g then is necessarily monic. Dually, a coequaliser for f and g is necessarily epic.*

Proof. Suppose that $a, b : X \rightrightarrows E$ are a parallel pair of morphisms such that $ea = eb$. We need to show that $a = b$. Now ea is a morphism from X to A and, since e is an equaliser, it factors through e uniquely. Since ea and eb are both factorisations of ea through e, we must have that $a = b$ for them to be the same factorisation. □

Definition 2.15 ($\star$)**.** Let $\mathcal{C}$ be a category and $f : A \rightarrowtail B$ a monomorphism.

1. We say that f is **regular** if it occurs as an equaliser for some two morphisms.

2. We say that f is **split** if there exists a morphism $g : B \to A$ such that $gf = 1_A$.

Proposition 2.16 ($\star$)**.** *Let $\mathcal{C}$ be a category and $f \in \operatorname{mor} \mathcal{C}$ a morphism.*

1. *If f is split monic then it is regular.*

2. *If f is epic and regular monic then it is an isomorphism.*

Proof.

Part 1: Suppose that $f : A \rightarrowtail B$ is split. Let g be its left-inverse so that $gf = 1_A$. We claim that f is the equaliser of fg and 1_B. Indeed, we have that $fgf = f1_A = 1_B f$. Now suppose that $h : C \to B$ satisfies $fgh = 1_B h = h$. Hence h factors through f via gh. Now suppose $fk = h$ is another factorisation of h through f. Then $fgh = fk$ whence $gh = k$ by the monicity of f. Hence h factors through f uniquely and f is an equaliser for fg and 1_B.

Part 2: Suppose that $f : A \to B$ is epic and regular. First assume that f is an equaliser of the maps $x, y : B \rightrightarrows C$. Then $xf = yf$ whence $x = y$ by epicness of f. Note that $x1_B = y1_B$ and so 1_B factors through f uniquely, say $fk = 1_B$ for some $k : B \to A$. Hence f is split epic.

By the dual of Part 1, f is regular epic and so f is both monic and split epic. By the dual of the previous paragraph, there exists a left inverse for f, say $lf = 1_A$ for some $l : B \to A$. Now observe that

$$l = l1_B = lfk = 1_A k = k$$

and so f must be an isomorphism.

□

Definition 2.17. Let $\mathcal{C}$ be a category and $\mathcal{G}$ a collection of objects of $\mathcal{C}$.

1. We say that $\mathcal{G}$ is a **separating family** if for all parallel pairs $f, g : A \rightrightarrows B$ in $\mathcal{C}$ such that $fh = gh$ for all $h : G \to A$ with $G \in \mathcal{G}$ we have $f = g$.

2. We say that $\mathcal{G}$ is a **dectecting family** if for all $f : A \to B$ such that every $h : G \to B$ with $G \in \mathcal{G}$ factors through f uniquely we have that f is an isomorphism.

3. If $\mathcal{G} = \{ G \}$ is a singleton, we say that G is a seperator/detector.

Definition 2.18. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F : \mathcal{C} \to \mathcal{D}$ a functor. We say that F **reflects isomorphisms** if for all $f \in \operatorname{mor} \mathcal{C}$ such that Ff is an isomorphism, we have that f is an isomorphism.

Lemma 2.19. *Let $\mathcal{C}$ be a locally small category and $G \in \mathcal{G}$ a collection of objects of $\mathcal{C}$. Then*

1. *$\mathcal{G}$ is a separating family if and only if $\mathcal{C}(G, \cdot)$ is faithful for all $G \in \mathcal{G}$.*

2. *$\mathcal{G}$ is a detecting family if and only if $\mathcal{C}(G, \cdot)$ reflects isomorphisms for all $G \in \mathcal{G}$.*

Proof.

Part 1: This is immediate from the definitions.

Part 2: Suppose that $\mathcal{G}$ is a detecting family. Let $f : A \to B$ be a morphism such that $\mathcal{C}(G, f)$ is an isomorphism. We need to show that f is an isomorphism. Let $h : G \to B$ be the morphism $\mathcal{C}(G, f)$. Then, by definition, we have that $h = f \circ g$ for some unique $g : G \to A$. Since $\mathcal{G}$ is a detecting family, we must have that f is an isomorphism and we are done. All conditions in the proof are necessary and sufficient so this also proves the backwards implication.

□

Proposition 2.20. *Let $\mathcal{C}$ be a category.*

1. *If $\mathcal{C}$ is balanced then any separating family is detecting.*

2. *If $\mathcal{C}$ has equalisers then every detecting family is separating.*

Proof.

Part 1: Let $\mathcal{G}$ be a separating family. Suppose that $f : A \to B$ is a morphism such that every $h : G \to B$ with $G \in \mathcal{G}$ factors through f uniquely. We must show that f is epic and monic. Assume that $hf = gf$ for some $g, h : B \to \mathcal{C}$. Then any $k : G \to B$ with $G \in \mathcal{G}$ satisfies $hk = gk$. Now, $\mathcal{G}$ is a separating family which implies that $h = g$ and so f is epic.

Now suppose that $fu = fv$ for some $u, v : D \rightrightarrows A$. Then for any $w : G \to D$ we have $fuw = fvw$. uw and vw are both factorisations of fuw through f and so must be equal. Since $\mathcal{G}$ is separating, it follows that $u = w$ and f is monic.

Since $\mathcal{C}$ is balanced and f is both monic and epic, we must have that f is an isomorphism and we are done.

Part 2: Let $\mathcal{G}$ be a detecting family. Suppose that $f, g : A \rightrightarrows B$ satisfies $fh = gh$ for all $h : G \to A$ with $G \in \mathcal{C}$. We need to show that $f = g$. Let $e : E \to A$ be an equaliser of f and g. Then every $h : G \to A$ with $fh = gh$ factors uniquely through e. Since $\mathcal{G}$ is a detecting family, it follows that e is an isomorphism and so $f = g$ as desired.

□

Example 2.21. Given a category $\mathcal{C}$, $\text{ob}\,\mathcal{C}$ is always a detecting and separating family. Indeed, to see that $\text{ob}\,\mathcal{C}$ is a detecting family, fix an $A \xrightarrow{f} B \in \text{mor}\,\mathcal{C}$ such that every $h : G \to B$ factors through f uniquely for every $G \in \text{ob}\,\mathcal{C}$. Then there exists $h : B \to A$ such that $fh = 1_B$. Note that $f1_A = f$ and $f = fhf$. By uniqueness of factorisations, we must have that $hf = 1_A$ and so f is an isomorphism and $\text{ob}\,\mathcal{C}$ is a detecting family.

To see that $\text{ob}\,\mathcal{C}$ is a separating family, fix a parallel pair $f, g : A \rightrightarrows B$ in $\mathcal{C}$ such that $fh = gh$ for all $h : G \to A$ with $G \in \mathcal{C}$. We need to show that $f = g$. By hypothesis, $id_A : A \to A$ satisfies $f1_A = g1_A$ and so $f = g$ as desired and $\text{ob}\,\mathcal{C}$ is a separating family.

Example 2.22. In **Set**, 1 is both a detector and a separator. Indeed, **Set** is a locally small category and $\mathbf{Set}(1, \cdot)$ is naturally isomorphic to the identity functor which is clearly faithful and so 1 is a separator. Furthermore, **Set** is balanced and so 1 is also a detector.

Example 2.23. In **Grp**, $\mathbb{Z}$ is both a detector and a separator. Indeed, **Grp** is a locally small category and $\mathbf{Grp}(\mathbb{Z}, \cdot)$ is naturally isomorphic to the forgetful functor which is clearly faithful and so 1 is a separator. Furthermore, **Grp** is balanced and so 1 is also a detector.

Definition 2.24 ($\star$)**.** Let $\mathcal{C}$ be a category and $P \in \text{ob}\,\mathcal{C}$ an object. We say that P is **projective** if given a diagram

$$\begin{array}{ccc} & & P \\ & \overset{g}{\swarrow} & \downarrow f \\ A & \overset{e}{\twoheadrightarrow} & B \end{array}$$

there exists $g : P \to A \in \text{mor}\,\mathcal{C}$ completing the diagram.

Lemma 2.25. *Let $\mathcal{C}$ be a locally small category and $P \in \text{ob}\,\mathcal{C}$ an object. Then P is projective if and only if $\mathcal{C}(P, \cdot)$ preserves epimorphisms.*

Proof. Suppose that P is projective. Fix an epimorphism $e : A \twoheadrightarrow B$ in $\mathcal{C}$. We need to show that $\mathcal{C}(P, e)$ is an epimorphism. Note that $\mathcal{C}(P, \cdot)$ has its image in **Set** so it suffices to show that $\mathcal{C}(P, e)$ is a surjective function of sets. Recall that $\mathcal{C}(P, e)$ is the function

$$\mathcal{C}(P, e) : \mathcal{C}(P, A) \to \mathcal{C}(P, B)$$
$$g \mapsto eg$$

Fix an $f \in \mathcal{C}(P, B)$. Since P is projective, there exists a $g \in \mathcal{C}(P, A)$ such that $f = eg$ which is exactly what it means for $\mathcal{C}(P, e)$ to be surjective. Note that the conditions throughout the proof are all necessary and sufficient and so the backwards implication is also proven. □

Proposition 2.26. *Let $\mathcal{C}$ be a locally small category. Then the representable functors $F : \mathcal{C} \to$ **Set** are η-projective in $[\mathcal{C}, \mathbf{Set}]$ where η is the class of pointwise surjective natural transformations.*

Proof. Suppose we are given a diagram

$$\begin{array}{ccc} & & \mathcal{C}(A, \cdot) \\ & & \downarrow \beta \\ F & \overset{\alpha}{\twoheadrightarrow} & G \end{array}$$

for some object $A \in \mathcal{C}$, functors $F, G : \mathcal{C} \rightrightarrows$ **Set** and natural transformations α and β. By the Yoneda Lemma, there is a one-to-one correspondence between elements of GA and natural transformations between $\mathcal{C}(A, \cdot)$ and G. Let $y \in GA$ be in correspondence with β. By pointwise surjectivity of α, there exists $x \in FA$ such that $\alpha_A(x) = y$. Appealing again to the Yoneda Lemma, there exists a $\gamma : \mathcal{C}(A, \cdot) \to F$ in correspondence with x. By naturality of the Yoneda Lemma, we then have that $\beta = \alpha\gamma$ and so $\mathcal{C}(A, \cdot)$ is projective. □

3 Adjunctions

In order to define adjunctions, we will need to expand the idea of hom-sets to include the possibility that a category is not locally small in the usual sense. In order to do this, we will need to expand the underlying set theory that we are working in.

Definition 3.1. Let U be a set. We say that U is a **Grothendieck universe** if the following conditions hold:

1. For all $x \in y \in U$ we have $x \in U$.
2. For all $x, y \in U$ we have $\{ x, y \} \in U$.
3. If $x \in U$ then the power set of x is a member of U.
4. If $\{ x_\alpha \}_{\alpha \in I}$ is a family of elements of U then $\bigcup_{\alpha \in I} x_\alpha$ is an element of U.

If $x \in U$ then we shall say that x is **U-small**.

Given a universe U, it can be shown that U is a model of ZFC (in other words all standard ZFC operations apply to the elements of U). As such, ZFC cannot prove the existence of a universe containing $\mathbb{N}$ and so, in order to obtain uncountable universes, we add the following axiom to ZFC:

For any set x there exists a universe U such that $x \in U$.

Thanks to this axiom, given any operation that lands outside of a universe, there is guaranteed to be a bigger universe in which that operation lands. Intuitively, this means that all sets are small given a large enough universe.

We may now apply this to categories. Given a universe, call a category U-small if $\operatorname{ob} \mathcal{C}$ and $\operatorname{mor} \mathcal{C}$ are U-small sets. Furthermore, call a category locally U-small if, given objects A and B, the collection of all morphisms between A and B form a U-small set. It is in this way that we can make sense of hom-sets for categories that are not locally small in the usual sense.

Given a category $\mathcal{C}$, choose a universe U such that the morphisms of $\mathcal{C}$ form a U-small set. Let V be a set large enough to contain all subsets of $\operatorname{mor} \mathcal{C}$. Define a category $\mathbf{Ens} = \mathbf{Set}_V$ where the objects of **Ens** are all the elements of V and the morphisms are all the functions between them. Then each hom-set $\mathcal{C}(A, B)$ with $A, B \in \mathcal{C}$ is an object of **Ens**. This then defines a covariant hom-functor

$$\begin{aligned} \mathcal{C}(A, \cdot) : \mathcal{C} &\to \mathbf{Ens} \\ B &\mapsto \mathcal{C}(A, B) \end{aligned}$$

Hence if a category $\mathcal{C}$ is not locally small in the usual sense, we may choose a universe U in which $\mathcal{C}$ is locally U-small and work with **Ens** instead of **Set**. Note that if V is chosen to be the universe of all small sets (in the usual sense) then $\mathbf{Ens} = \mathbf{Set}$.

Henceforth, we shall assume that all categories are locally small up to a large enough universe U.

Definition 3.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ functors. By an **adjunction** between F and G, we mean a natural isomorphism between the functors

$$\begin{aligned} \mathcal{C}(\cdot, G(\cdot)) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} &\to \mathbf{Set} \\ (A, B) &\mapsto \mathcal{C}(A, GB) \end{aligned}$$

$$\begin{aligned} \mathcal{D}(F(\cdot), \cdot) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} &\to \mathbf{Set} \\ (A, B) &\mapsto \mathcal{D}(FA, B) \end{aligned}$$

In this case, we say that F is **left adjoint** to G and write $F \dashv G$.

Example 3.3. Let $F : \mathbf{Set} \to \mathbf{Grp}$ be the free group functor which sends a set A to the free group on A. Then F is left adjoint to the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$. Indeed fix a set A and a group G and consider the mapping $\Phi : \mathbf{Set}(A, UG) \to \mathbf{Grp}(FA, G)$ which takes a function $A \xrightarrow{f} UG$ and sends it to the homomorphism $FA \xrightarrow{\varphi_f} G$ defined by taking the values of f on the basis elements of FA then extending linearly. It is readily shown that Φ is a bijection and so we have a one-to-one correspondence between the two hom-sets.

To see that this correspondence is natural in both A and G, let $\alpha_{A,G}$ be the bijection defined above for some set A and group G. Fix some morphism $g : A \to A'$ in $\mathbf{Set}^{\mathrm{op}}$. For naturality in A, we need to show that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Grp}(FA, G) & \xrightarrow{\mathbf{Grp}(Fg,G)} & \mathbf{Grp}(FA', G) \\ {\scriptstyle \alpha_{A,G}} \uparrow & & {\scriptstyle \alpha_{A',G}} \uparrow \\ \mathbf{Set}(A, UG) & \xrightarrow{\mathbf{Set}(g,UG)} & \mathbf{Set}(A', UG) \end{array}$$

Let $f \in \mathbf{Set}(A, UG)$. On one hand we have

$$\alpha_{A',G}(\mathbf{Set}(g, UG)(f)) = \alpha_{A',G}(f \circ g^{\mathrm{op}}) = \varphi_{f \circ g^{\mathrm{op}}}$$

On the other hand, note that Fg is the unique group homomorphism between FA' and FA induced by g^{op} which is exactly $\varphi_{g^{\mathrm{op}}}$. Hence

$$\mathbf{Grp}(Fg, G)(\alpha_{A,G}(f)) = \mathbf{Grp}(\varphi_{g^{\mathrm{op}}}, G)(\varphi_f) = \varphi_f \circ \varphi_{g^{\mathrm{op}}} = \varphi_{f \circ g^{\mathrm{op}}}$$

We thus see that the diagram is commutative and α is natural in A. Naturality in G follows in a similar way.

Example 3.4. Let $D : \mathbf{Set} \to \mathbf{Top}$ be the functor endowing a set with the discrete topology. Then D is left adjoint to the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$. Indeed, fix a set A and a topological space τ. Consider the mapping $\Phi : \mathbf{Set}(A, U\tau) \to \mathbf{Top}(DA, \tau)$ which sends a function $A \xrightarrow{f} U\tau$ to the corresponding continuous map $DA \xrightarrow{f} \tau$ (this is well-defined since the inverse image of an open set of τ under f will always be open in DA). Note that we are abusing notation with f referring both to a function sets and a continuous map of topological spaces. As before, it is easy to see that Φ is a one-to-one correspondence between the hom-sets.

Let $\alpha_{A,\tau}$ be the correspondence given above. We shall show that α is natural in A. To this end, fix some morphism $g : A \to A'$ in $\mathbf{Set}^{\mathrm{op}}$. We need to show that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Top}(DA,\tau) & \xrightarrow{\mathbf{Top}(Dg,\tau)} & \mathbf{Top}(DA',\tau) \\ {\scriptstyle \alpha_{A,\tau}}\uparrow & & {\scriptstyle \alpha_{A',\tau}}\uparrow \\ \mathbf{Set}(A,U\tau) & \xrightarrow{\mathbf{Set}(g,U\tau)} & \mathbf{Set}(A',U\tau) \end{array} \quad \text{exact}$$

Let $f \in \mathbf{Set}(A, UG)$. On one hand, we have

$$\alpha_{A',\tau}(\mathbf{Set}(g, U\tau)(f)) = \alpha_{A',\tau}(f \circ g^{\mathrm{op}}) = f \circ g^{\mathrm{op}}$$

On the other hand we have

$$\mathbf{Top}(Dg, \tau)(\alpha_{A,\tau}(f)) = \mathbf{Top}(g^{\mathrm{op}}, \tau)(f) = f \circ g^{\mathrm{op}}$$

and so α is natural in A. Naturality in τ follows in a similar way.

Example 3.5. Let $I : \mathbf{Set} \to \mathbf{Top}$ be the functor endowing a set with the indiscrete topology. Then I is right adjoint to the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$.

Example 3.6. Consider the functor $D : \mathbf{Set} \to \mathbf{Cat}$ which takes a set A to the discrete category DA where $\operatorname{ob} DA = A$ and $\operatorname{mor} DA = \{ 1_X : X \to X \mid X \in A \}$. Then D is left adjoint to $\operatorname{ob} : \mathbf{Cat} \to \mathbf{Set}$. On the other hand, the functor $I : \mathbf{Set} \to \mathbf{Cat}$ such that $\operatorname{ob} IA = A$ and $\operatorname{mor} IA = \{ f : X \to Y \mid X, Y \in A \}$ is a right adjoint to ob.

Example 3.7. The contravariant powerset functor $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is self-adjoint on the right. This is easy to see by arguing with cardinal numbers. We need to find a bijection between $\mathbf{Set}(A, P^*B)$ and $\mathbf{Set}(B, P^*A)$. Assume that A and B are infinite sets of cardinality κ and μ respectively with $\kappa \geq \mu$ (the finite case follows the same argumentation). Then

$$|\mathbf{Set}(A, P^*B)| = (2^\mu)^\kappa = 2^{\mu\kappa} = 2^\kappa$$

On the other hand, we have that

$$|P^*(A \times B)| = 2^{\kappa\mu} = 2^\kappa$$

The two sets thus have the same cardinality and so there must exist a bijection between them. We then have that

$$\mathbf{Set}(A, P^*B) \cong P^*(A \times B) \cong P^*(B \times A) \cong \mathbf{Set}(B, P^*A)$$

It is also easy to see that this bijection is natural in both A and B.

Definition 3.8 ($\star$)**.** Let $\mathcal{C}$ be a category. We define an **initial object** $I \in \operatorname{ob}\mathcal{C}$ to be one such that for all $X \in \operatorname{ob}\mathcal{C}$ there is exactly one morphism $I \to X$. Dually, a **terminal object** $T \in \operatorname{ob}\mathcal{C}$ is one such that for all $X \in \mathcal{C}$, there is exactly one morphism $X \to T$.

Example 3.9. The terminal object of **Cat** is the discrete category consisting of one object and one morphism.

Example 3.10. Let 1 be the terminal object of **Cat** and $\mathcal{C}$ a category. Consider the functor $F : \mathcal{C} \to 1$ and suppose that $L \dashv F$. Let A be the unique object of $\mathcal{C}$ such that $LX = A$. By definition, we have

$$\mathcal{C}(A, LX) \cong 1(FA, X) = 1(X, X) = \{ 1_X \}$$

and so A is an initial object of $\mathcal{C}$. In other words, specifying a left adjoint for F is the same as specifying an inital object of $\mathcal{C}$. Dually, specifying a right adjoint for F is the same as specifying a terminal object of $\mathcal{C}$.

Definition 3.11 ($\star$)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $G : \mathcal{D} \to \mathcal{C}$ a functor. Given $A \in \operatorname{ob}\mathcal{C}$, we define the **comma** (or **arrow**) category, denoted $(A \downarrow G)$, to be the category whose objects are pairs (B, f) with $B \in \operatorname{ob}\mathcal{D}$ and $f : A \to GB$ and whose morphisms $(B, f) \to (B', f')$ are morphisms $(B \xrightarrow{g} B') \in \operatorname{mor}\mathcal{D}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ & {\scriptstyle f'}\searrow & \downarrow{\scriptstyle Gg} \\ & & GB' \end{array}$$

commutes.

Theorem 3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $G : \mathcal{D} \to \mathcal{C}$ a functor. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for all $A \in \mathcal{C}$.*

Proof. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint for G. In particuar, we have that $\mathcal{D}(FA, FA) \cong \mathcal{C}(A, GFA)$ for any $A \in \operatorname{ob}\mathcal{C}$. Hence $(FA \xrightarrow{1_{FA}} FA) \in \operatorname{mor}\mathcal{D}$ corresponds to some morphism $(A \xrightarrow{\eta_A} GFA) \in \operatorname{mor}\mathcal{C}$. We claim that (FA, η_A) is an initial object of $(A \downarrow G)$. To this end, fix an object (B, f) in $(A \downarrow G)$. We need to show that there exists a unique morphism $(FA, \eta_A) \to (B, f)$. By definition, we need to find a morphism $(FA \xrightarrow{g} B) \in \operatorname{mor}\mathcal{D}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle Gg} \\ & & GB \end{array}$$

commutes. Were such a g to exist, the above diagram commutes if and only if the diagram

$$\begin{array}{ccc} FA & \xrightarrow{1_A} & FA \\ & {\scriptstyle f'}\searrow & \downarrow{\scriptstyle g} \\ & & B \end{array}$$

commutes where f' corresponds to f under the adjunction. Hence g must be equal to f' and we are done.

Conversely, suppose that for all $A \in \operatorname{ob}\mathcal{C}$ we are given an initial object (B_A, η_A) of $(A \downarrow G)$. We define a functor F by $FA = B_A$ on objects and on morphisms $f : A \to A'$ to be the unique morphism in $(A \downarrow g)$ between (FA, η_A) and $(FA', \eta_{A'} f)$. In other words, Ff is the unique morphism in $\mathcal{D}$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

To see that F is indeed functorial, suppose that $f : A \to A'$ and $f' : A' \to A''$ are morphisms in $\mathcal{C}$. Then $(Ff')(Ff)$ and $F(f'f)$ are both morphisms between (FA, η_A) and $(FA'', \eta_{A''} f'f)$. But (FA, η_A) is initial so these morphisms must be the same.

We claim that F is left-adjoint to G. Note that, by construction, $\eta : 1_{\mathcal{C}} \to GF$ is a natural transformation. Suppose we are given an object (B, y) in $(A \downarrow g)$. Then y is a morphism $y : A \to GB$. Since (FA, η_A) is initial, there exists a unique morphism $(FA, \eta_A) \to (B, y)$ in $(A \downarrow g)$. Such a morphism is a morphism $(FA \xrightarrow{x} B) \in \operatorname{mor}\mathcal{D}$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & \searrow^{y} & \downarrow{\scriptstyle Gx} \\ & & GB \end{array}$$

This gives us a bijection between $\mathcal{C}(A, GB)$ and $\mathcal{D}(FA, B)$. Write this bijection as

$$\begin{aligned} \alpha_{A,B} : \mathcal{D}(FA, B) &\to \mathcal{C}(A, GB) \\ x &\mapsto (Gx)\eta_A \end{aligned}$$

We claim that α is a natural isomorphism between the functors

$$\begin{aligned} \mathcal{C}(\cdot, G(\cdot)) : \mathcal{C}^{\text{op}} \times \mathcal{D} &\to \mathbf{Set} \\ (A, B) &\mapsto \mathcal{C}(A, GB) \end{aligned}$$

$$\begin{aligned} \mathcal{D}(F(\cdot), \cdot) : \mathcal{C}^{\text{op}} \times \mathcal{D} &\to \mathbf{Set} \\ (A, B) &\mapsto \mathcal{D}(FA, B) \end{aligned}$$

To this end, we must show that α is natural in both A and B. Hence suppose that $(A' \xrightarrow{f} A) \in \operatorname{mor} \mathcal{C}$. We must show that the diagram

$$\begin{array}{ccc} \mathcal{C}(A, GB) & \xrightarrow{\mathcal{C}(f, GB)} & \mathcal{C}(A', GB) \\ {\scriptstyle \alpha_{A,B}}\uparrow & & {\scriptstyle \alpha_{A',B}}\uparrow \\ \mathcal{D}(FA, B) & \xrightarrow{\mathcal{D}(F, B)} & \mathcal{D}(FA', B) \end{array}$$

commutes. To see this, let $x \in \mathcal{D}(FA, B)$. By naturality of η, we have

$$\begin{aligned} \alpha_{A',B}(\mathcal{D}(Ff, B)(x)) &= \alpha_{A',B}(x \circ (Ff)) \\ &= G(x \circ (Ff))\eta_{A'} \\ &= Gx \circ GFf \circ \eta_{A'} \\ &= Gx \circ \eta_A \circ f \\ &= \alpha_{A,B}(x) \circ f \\ &= \mathcal{C}(f, GB)(\alpha_{A,B}(x)) \end{aligned}$$

and so α is natural in A. For naturality in B, suppose that $(B \xrightarrow{f} B') \in \operatorname{mor} \mathcal{D}$. We must show that the diagram

$$\begin{array}{ccc} \mathcal{C}(A, GB) & \xrightarrow{\mathcal{C}(A, Gf)} & \mathcal{C}(A, GB') \\ {\scriptstyle \alpha_{A,B}}\uparrow & & {\scriptstyle \alpha_{A,B'}}\uparrow \\ \mathcal{D}(FA, B) & \xrightarrow{\mathcal{D}(FA, f)} & \mathcal{D}(FA, B') \end{array}$$

To see this, let $x \in \mathcal{D}(FA, B)$. We have that

$$\alpha_{A,B'}(\mathcal{D}(FA, f)(x)) = \alpha_{A,B'}(fx) = G(fx)\eta_A = Gf \circ Gx \circ \eta_A = Gf \circ \alpha_{A,B}(x) = \mathcal{C}(A, Gf)(\alpha_{A,B}(x))$$

and so α is natural in B. Therefore, F is a left adjoint for G. □

Corollary 3.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. If $G : \mathcal{D} \to \mathcal{C}$ is a functor with left adjoints $F, F' : \mathcal{C} \to \mathcal{D}$ then there exists a natural isomorphism $\alpha : F \to F'$.*

Proof. By Theorem 3.12 (FA, η_A) and $(F'A, \eta'_A)$ are both initial objects of $(A \downarrow G)$ for all $A \in \operatorname{ob}\mathcal{C}$. It is easy to see that any two inital objects of a category must be isomorphic so there exists an isomorphism $(FA, \eta_A) \xrightarrow{\alpha_A} (F'A, \eta'_A) \in \operatorname{mor}(A \downarrow G)$. By definition, α_A is a morphism $(FA \xrightarrow{\alpha_A} F'A) \in \operatorname{mor}\mathcal{D}$. We claim that α is natural in A. Indeed, given $f : A \to A$ we have the diagram

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & F'A \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle F'f} \\ FA' & \xrightarrow{\alpha_A} & F'A' \end{array}$$

Note that both ways round the square are morphisms $(FA, \eta_A) \to (F'A', \eta'_{A'}f)$ in $(A \downarrow G)$. But (FA, η_A) is initial in $(A \downarrow G)$ and so these morphisms must be equal and the diagram is commutative. Hence α is natural in A and is a natural isomorphism $\alpha : F \to F'$. □

Proposition 3.14. *Suppose that we are given functors*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathcal{E}$$

such that $F \dashv G$ and $H \dashv K$. Then $HF \dashv GK$.

Proof. Suppose $A \in \mathcal{C}$ and $B \in \mathcal{E}$. By definition, we have bijections

$$\mathcal{E}(HFA, C) \cong \mathcal{D}(FA, KC) \cong \mathcal{C}(A, GKC)$$

The result then follows upon composing these bijections and realising the resulting bijection is natural in both A and C. □

Corollary 3.15. *Suppose that we are given a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle H} \\ \mathcal{E} & \xrightarrow{K} & \mathcal{F} \end{array}$$

of categories and functors between them. If all the functors in the diagram have left adjoints then the diagram

$$\begin{array}{ccc} \mathcal{C} & \longleftarrow & \mathcal{D} \\ \uparrow & & \uparrow \\ \mathcal{E} & \longleftarrow & \mathcal{F} \end{array}$$

of left adjoints also commutes up to natural isomorphism.

Proof. Suppose that $F' \dashv F, G' \dashv G, H' \dashv H$ and $K' \dashv K$ so that the completed diagram is

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow[F']{} & \mathcal{D} \\ {\scriptstyle G'}\uparrow & & {\scriptstyle H'}\uparrow \\ \mathcal{E} & \xleftarrow[K']{} & \mathcal{F} \end{array}$$

By Proposition 3.14, $F'H' \dashv HF$ and $G'K' \dashv KG$. But $HF = KG$ and so the functor has two left adjoints $F'H'$ and $G'K'$. By Corollary 3.13, there must exist a natural isomorphism between $F'H'$ and $G'K'$ so we are done. □

Remark. Suppose we are given functors

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \underset{H}{\overset{G}{\rightrightarrows}} \mathcal{C} \xrightarrow{K} \mathcal{D}$$

and a natural transformation $\alpha : G \to H$. Then we can define the following natural transformations:

- $\alpha_F : GF \to HF$ with components given by $(\alpha_F)_A = \alpha_{FA}$ for all $A \in \operatorname{ob} \mathcal{A}$.
- $K\alpha : KG \to KH$ with components given by $(K\alpha)_B = K\alpha_B$ for all $B \in \operatorname{ob} \mathcal{B}$.

Theorem 3.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $G : \mathcal{D} \to \mathcal{C}$ a functor. Then specifying a left adjoint $F : \mathcal{C} \to \mathcal{D}$ for G is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{D}}$ such that the diagrams*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & {\scriptstyle 1_F}\searrow & \downarrow{\scriptstyle \varepsilon_F} \\ & & F \end{array} \qquad\qquad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & {\scriptstyle 1_G}\searrow & \downarrow{\scriptstyle G\varepsilon} \\ & & G \end{array}$$

commute. In this case, we say that η and ε are the ***unit*** *and* ***counit*** *of the adjunction $F \dashv G$ respectively and they satisfy the* ***triangular identities****.*

Proof. Suppose we are given an adjunction $F \dashv G$. Let $\alpha_{A,B} : \mathcal{D}(FA, B) \to \mathcal{C}(A, GB)$ be the correspondence between the hom-sets. Define $\eta_A = \alpha_{A,FA}(1_{FA})$ and $\varepsilon_B : \alpha^{-1}_{GB,B}(1_{GB})$. We claim that η and ε are the desired natural transformations. We first show that η and ε are natural in A and B respectively. It suffices to show this for η as ε is dual.

Fix $f : A \to A'$, we need to show that the diagram

$$\begin{array}{ccc} GFA & \xrightarrow{GFf} & GFA' \\ {\scriptstyle \eta_A}\uparrow & & \uparrow{\scriptstyle \eta_{A'}} \\ A & \xrightarrow{f} & A' \end{array}$$

commutes. By naturality of α, the following two squares commute

$$\begin{array}{ccc} \mathcal{D}(FA, FA) & \xrightarrow{\mathcal{D}(FA,Ff)} & \mathcal{D}(FA, FA') \\ \downarrow{\scriptstyle \alpha_{A,FA}} & & \downarrow{\scriptstyle \alpha_{A,FA'}} \\ \mathcal{C}(A, GFA) & \xrightarrow{\mathcal{C}(A,GFf)} & \mathcal{C}(A, GFA') \end{array} \qquad \begin{array}{ccc} \mathcal{D}(FA', FA') & \xrightarrow{\mathcal{D}(Ff,FA')} & \mathcal{D}(FA, FA') \\ \downarrow{\scriptstyle \alpha_{A',FA'}} & & \downarrow{\scriptstyle \alpha_{A,FA'}} \\ \mathcal{C}(A', GFA') & \xrightarrow{\mathcal{C}(f,GFA')} & \mathcal{C}(A, GFA') \end{array}$$

Chasing 1_{FA} and $1_{FA'}$ around these diagrams gives

$$\begin{array}{ccc} 1_{FA} & \longmapsto & Ff \\ \downarrow & & \downarrow \\ \eta_A & \longmapsto & GF(f) \circ \eta_A = \alpha_{A,FA'}(Ff) \end{array} \qquad \begin{array}{ccc} 1_{FA'} & \longmapsto & Ff \\ \downarrow & & \downarrow \\ \eta_{A'} & \longmapsto & \eta_{A'} \circ f = \alpha_{A,FA'}(Ff) \end{array}$$

Combining these results gives $GF(f) \circ \eta_A = \eta_{A'} \circ f$ and so η is natural in A.

We must now check the triangular identities. Again, we shall show that one of the identities holds, the other follows dually. First fix $A \in \operatorname{ob} \mathcal{C}$

$$1_{FA} = \alpha^{-1}_{A,FA}(\alpha_{A,FA}(1_{FA}) = \alpha^{-1}_{A,FA}(\eta_A) = \alpha^{-1}_{A,FA}(1_{GFA} \circ \eta_A)$$

Now suppose that $g : Z' \to Z$ is a morphism in $\mathcal{C}$. By naturality of α in A the diagram

$$\begin{array}{ccc} \mathcal{C}(Z,GB) & \xrightarrow{\mathcal{C}(g,GB)} & \mathcal{C}(Z',GB) \\ \downarrow{\scriptstyle \alpha^{-1}_{Z,B}} & & \downarrow{\scriptstyle \alpha^{-1}_{Z',B}} \\ \mathcal{D}(FZ,B) & \xrightarrow{\mathcal{D}(Fg,B)} & \mathcal{D}(FZ',B) \end{array}$$

commutes. Hence given $f : Z \to GB$ we have

$$\alpha^{-1}_{Z',B}(f \circ g) = \alpha^{-1}_{Z,B}(f) \circ (Fg)$$

Now let $Z' = A, Z = GFA, B = FA, f = 1_{GFA}$ and $g = \eta_A : A \to GFA$. Then

$$\alpha^{-1}_{A,FA}(1_{GFA} \circ \eta_A) = \varepsilon_{FA} \circ F\eta_A$$

and so $1_F = \varepsilon_F \circ F\eta$ and the triangular identity is satisfied.

Conversely, suppose that we are given natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{D}}$ satisfying the triangular identities. Given $A \in \operatorname{ob}\mathcal{C}$ and $B \in \operatorname{ob}\mathcal{D}$, we need to find a bijection $\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B)$ which is natural in both A and B.

We define $\Phi : \mathcal{C}(A, GB) \to \mathcal{D}(FA, B)$ by $\Phi(A \xrightarrow{f} GB) = FA \xrightarrow{Ff} FGB \xrightarrow{\varepsilon_B} B$. Similarly, define $\Psi : \mathcal{D}(FA, B)$ by $\Psi(FA \xrightarrow{g} B) = A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$. We claim that Φ and Ψ are mutually inverse. Fix $f \in \mathcal{C}(A, GB)$. By the triangular identities and the naturality of η, we have

$$\Psi(\Phi(f)) = \Psi(\varepsilon_B \circ (Ff)) = G(\varepsilon_B \circ Ff) \circ \eta_A = G\varepsilon_B \circ GFf \circ \eta_A = G\varepsilon_B \circ \eta_{GB} \circ f = f$$

The other composition follows similarly. To show naturality in A, fix some $f : A' \to A$. We need to show that the diagram

$$\begin{array}{ccc} \mathcal{C}(A,GB) & \xrightarrow{\mathcal{C}(f,GB)} & \mathcal{C}(A',GB) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathcal{D}(FA,B) & \xrightarrow{\mathcal{D}(Ff,B)} & \mathcal{D}(FA',B) \end{array}$$

Let $g \in \mathcal{C}(A, GB)$. We have that

$$\Phi(\mathcal{C}(f, GB)(g)) = \Phi(g \circ f) = \varepsilon_B \circ F(g \circ f) = \varepsilon_B \circ Fg \circ Ff = \mathcal{C}(Ff, B)(\varepsilon_B \circ Fg) = \mathcal{C}(Ff, B)(\Phi(g))$$

as desired. Naturality in B follows from considering a similar square and the naturality of ε. □

Proposition 3.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be equivalent categories with the equivalence given by functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$. Then there exist natural isomorphisms $\alpha' : 1_{\mathcal{C}} \to GF$ and $\beta' : FG \to 1_{\mathcal{D}}$ satisfying the triangular identities. In particular, $F \dashv G$.*

Proof. Let $\alpha' = \alpha$ and $\beta' = FG \xrightarrow{\beta^{-1}_{FG}} FGFG \xrightarrow{F\alpha^{-1}_G} FG \xrightarrow{\beta} 1_{\mathcal{D}}$. We must show that α' and β' satisfy the triangular identities. First note that

$$\beta'_F \circ F\alpha' = \beta'_F(F\alpha) = \beta_F \circ F\alpha^{-1}_{GF} \circ \beta^{-1}_{FGF} \circ F\alpha$$

By the naturality of β, the diagram

$$\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ \downarrow{\scriptstyle \beta_{FG}} & & \downarrow{\scriptstyle \beta} \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$$

commutes. Hence $\beta(FG\beta) = \beta(\beta_{FG})$. But β is monic and so $FG\beta = \beta_{FG}$. Similarly, $GF\alpha = \alpha_{GF}$. We then have

$$\begin{aligned} \beta'_F \circ F\alpha' = \beta'_F \circ F\alpha &= \beta_F \circ (F\alpha_{GF})^{-1} \circ \beta^{-1}_{FGF} \circ F\alpha \\ &= \beta_F \circ (FGF\alpha)^{-1} \circ (FGF\alpha) \circ F\alpha \\ &= \beta_F \circ (FGF\alpha)^{-1} \circ (FGF\alpha) \circ \beta^{-1}_F \\ &= 1_F \end{aligned}$$

For the other identity, we have

$$\begin{aligned} G\beta' \circ \alpha'_G &= G\beta \circ \alpha_G = G\beta \circ (GF\alpha_G)^{-1} \circ (G\beta FG)^{-1} \circ \alpha_G \\ &= G\beta \circ (\alpha_{GFG})^{-1} \circ (GFG\beta)^{-1} \circ \alpha_G \\ &= G\beta \circ (GFG\beta) \circ (GFG\beta)^{-1} \circ (G\beta)^{-1} \\ &= 1_G \end{aligned}$$

□

Proposition 3.18. *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be an adjoint pair with counit* $\varepsilon : FG \to 1_{\mathcal{D}}$. *Then*

1. *G is faithful if and only if ε is pointwise epic.*

2. *G is fully faithful if and only if ε is a natural isomorphism.*

Proof.

Part 1: Suppose that ε_B is epic for all $B \in \operatorname{ob} \mathcal{D}$. Let $g, g' : B \to B'$. We need to show that if $Gg = Gg'$ then $g = g'$. Now, Gb and Gb' correspond to $g\varepsilon_B$ and $g'\varepsilon_B$ respectively. We thus have $g\varepsilon_B = g'\varepsilon_B$. But ε_B is epic and so $g = g'$ and G is faithful. The conditions in the proof are all necessary and sufficient and so the backwards implication also follows.

Part 2: First suppose that ε_B is an isomorphism. By Part 1, G is faithful so it suffices to show that G is full. To this end, suppose that $g : GB \to GB'$ in $\mathcal{C}$, we need to exhibit an $f : B \to B'$ in $\mathcal{D}$ such that $Gf = g$. g corresponds to some $\overline{f} : FGB \to B'$ under the adjunction. Let $f = \overline{f}(\varepsilon_B)^{-1}$. Then

$$Gf = G\varepsilon_{B'} \circ GFg \circ G\varepsilon_B^{-1}$$

By naturality of ε, this equals g and we are done.

Conversely, suppose that G is full and faithful. Then $GB \xrightarrow{\eta_{GB}} GFGB$ is of the form Gh for some $B \xrightarrow{h} FGB$. By the triangular identities, we have that $(G\varepsilon_B)(\eta_{GB}) = 1_{GB}$ and so

$$G(1_B) = 1_{GB} = (G\varepsilon_B)(Gh) = G(\varepsilon_B h)$$

whence $1_{GB} = \varepsilon_B h$. Conversely, $h\varepsilon_B$ corresponds to η_{GB} under the adjunction. Passing back through the adjunction, η_{GB} corresponds to 1_{FGB} and so $h\varepsilon = 1_{FGB}$ from which it follows that ε_B is an isomorphism.

□

Definition 3.19 ($\star$)**.** Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjoint pair.

1. We say that $F \dashv G$ is a **reflection** if G is full and faithful.

2. By a **reflective** subcategory of $\mathcal{C}$, we mean a full subcategory $\mathcal{C}'$ for which the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint.

Example 3.20. Consider the inclusion functor $G : \mathbf{AbGrp} \to \mathbf{Grp}$. Then G has a left adjoint given by the functor $F : \mathbf{Grp} \to \mathbf{AbGrp}$ sending a group H to its Abelianisation H' where H' is the derived subgroup of G. Then the adjunction $F \dashv G$ is reflective and **AbGrp** is a reflective subcategory of **Grp**.

Example 3.21. Consider the inclusion functor $G : \mathbf{TfAbGrp} \to \mathbf{AbGrp}$. Then G has a left adjoint given by the functor $F : \mathbf{AbGrp} \to \mathbf{TfAbGrp}$ which sends an abelian group H to H/H_τ where H_τ is the torsion subgroup of H. Then the adjunction $F \dashv G$ is reflective and **TfAbGrp** is a reflective subgroup of **AbGrp**.

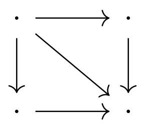
Similarly, consider the inclusion functor $G : \mathbf{TAbGrp} \to \mathbf{Abgrp}$. Then G has a right adjoint given by the functor $F : \mathbf{AbGrp} \to \mathbf{TAbGrp}$ which sends an abelian group H to its torsion subgroup H_τ. Then the adjunction $G \dashv F$ is coreflective and **TAbGrp** is a coreflective subgroup of **AbGrp**.

Example 3.22. Consider the inclusion functor $G : \mathbf{KHaus} \to \mathbf{Top}$. Then G has a left adjoint given by the functor $F : \mathbf{Top} \to \mathbf{KHaus}$ which sends a topological space X to its Stone-Čech compactification βX.[1]

4 Limits

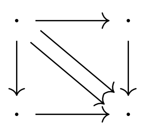
Definition 4.1. Let J be a category. By a **diagram of shape J** in $\mathcal{C}$, we mean a functor $D : J \to \mathcal{C}$.

Example 4.2. Let J be the finite category given by the following representation:



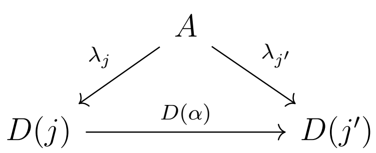
Then a **diagram of shape J** in $\mathcal{C}$ is a commutative square in $\mathcal{C}$. The objects $D(j)$ with $j \in \mathrm{ob}\, J$ are called the **vertices** of D and the morphisms $D(\alpha)$ with $\alpha \in \mathrm{mor}\, J$ are called the **morphisms** of D.

Example 4.3. Let J be the finite category given by the following representation:



Then a diagram of shape J in $\mathcal{C}$ is a square in $\mathcal{C}$ that is not necessarily commutative.

Definition 4.4 ($\star$)**.** Let J and $\mathcal{C}$ be categories and $D : J \to \mathcal{C}$ be a diagram. We define a **cone** over D to be a pair $(A, (\lambda_j | j \in \mathrm{ob}\, J))$ with $A \in \mathcal{C}$ and $\lambda_j : A \to D(j)$ such that



[1]Stone and Čech both gave different constructions of the compactification and the only way to show that they are the same is to show that they are left adjoint to the inclusion functor.

commutes for every $\alpha : j \to j'$ in J. We say that A is the **apex** and λ_i are the **legs** of the cone.

Lemma 4.5 ($\star$)**.** *Let J and $\mathcal{C}$ be categories and $A \in \operatorname{ob}\mathcal{C}$. If $\triangle A$ is the constant diagram whose vertices are all A and whose edges are all 1_A then cones over a diagram $D : J \to \mathcal{C}$ correspond to natural transformations $\alpha : \triangle A \to D$.*

Proof. Suppose we are given a diagram $D : J \to \mathcal{C}$ and a cone over D given by the pair $(B, (\lambda_j | j \in \operatorname{ob} J))$ for some $B \in \mathcal{C}$ and $\lambda_j : B \to D(j)$. Define a natural transformation $\alpha : \triangle A \to D$ by defining $\alpha_j : \triangle A(j) \to D(j)$ to be λ_j. Then the fact that the naturality square commutes follows immediately from the fact that $\triangle A(j) = \triangle A(j')$ for all $j, j' \in \operatorname{ob} J$ and the definition of a cone. The backwards implication follows from the same reasoning. $\square$

Definition 4.6 ($\star$)**.** Let $D : J \to \mathcal{C}$ be a diagram and $(A, (\lambda_j))$ and $(B, (\mu_j))$ cones over D. We define a **morphism of cones** $(A, (\lambda_j)) \to (B, \mu_j))$ to be a morphism $f : A \to B$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle\lambda_j}\searrow \qquad \swarrow{\scriptstyle\mu_j} & \\ & D(j) & \end{array}$$

commutes.

Definition 4.7 ($\star$)**.** Let $D : J \to \mathcal{C}$ be a diagram. We denote the category of all cones over D together with the morphisms between them as $\mathbf{Cone}(D)$.

Proposition 4.8 ($\star$)**.** *Let $D : J \to \mathcal{C}$ be a diagram and consider the functor*

$$\begin{aligned} \triangle : \mathcal{C} &\to [J, \mathcal{C}] \\ A &\mapsto \triangle A \end{aligned}$$

Then $\mathbf{Cone}(D) = (\triangle \downarrow D)$.

Proof. Suppose that $(B, f) \in \operatorname{ob}(\triangle \downarrow D)$. Then $B \in \operatorname{ob}\mathcal{C}$ and f is a natural transformation $f : \triangle B \to D$ which is the same as a cone over D. Suppose that $(B, f), (B', f') \in \operatorname{ob}(\triangle \downarrow D)$ and $g : (B', f) \to (B, f)$ is a morphism between them. Then g is a morphism $g : B' \to B$ in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} D & \xleftarrow{\;f\;} & \triangle B \\ & {\scriptstyle f'}\nwarrow & \uparrow{\scriptstyle \triangle g} \\ & & \triangle B' \end{array}$$

commutes. This diagram commutes if and only if for all $j \in \operatorname{ob} J$ the diagram

$$\begin{array}{ccc} D(j) & \xleftarrow{\;f_j\;} & B \\ & {\scriptstyle f'_j}\nwarrow & \uparrow{\scriptstyle g} \\ & & B' \end{array}$$

which is exactly the same as a morphism between the cones (B', f) and (B, f). $\square$

Definition 4.9 ($\star$)**.** Let $D : J \to \mathcal{C}$ be a diagram. Then a **limit** (dually **colimit**) of D is a terminal object of $\mathbf{Cone}(D)$ (dually an initial object of $\mathbf{Cocone}(D)$). We say a (co)limit of D is **finite** (respectively small) if J is finite (respectively small).

Definition 4.10 ($\star$)**.** Let J and $\mathcal{C}$ be categories. Then we say that $\mathcal{C}$ **has limits of shape J** if the functor $\triangle : \mathcal{C} \to [J, \mathcal{C}]$ has a right adjoint.

Proposition 4.11 ($\star$)**.** *Let J and $\mathcal{C}$ be categories. Then $\mathcal{C}$ has limits of shape J if and only if every diagram $D : J \to \mathcal{C}$ has a limit.*

Proof. By Theorem 3.12, the functor $\triangle : \mathcal{C} \to [J, \mathcal{C}]$ has a right adjoint if and only if for all $D \in \operatorname{ob} \mathcal{C}$ the category $(\triangle \downarrow D) = \mathbf{Cone}(D)$ has a terminal object. This is exactly what it means for a diagram $D : J \to \mathcal{C}$ to have a limit. □

Example 4.12. Suppose that $J = \varnothing$. Then for any category $\mathcal{C}$ there is a unique diagram $D : J \to \mathcal{C}$. It is easy to see that $(\triangle \downarrow D) \cong \mathcal{C}$ and so a limit (dually colimit) for D is a terminal object of $\mathcal{C}$ (dually initial object).

Example 4.13. Suppose J is the discrete category with two objects and $D : J \to \mathcal{C}$ is a diagram of shape J. Denote the two objects of this diagram by X_1 and X_2. Then a limit for D is an object $X_1 \times X_2$ equipped with morphisms $\lambda_1 : X_1 \times X_2 \to X_1$ and $\lambda_2 : X_1 \times X_2 \to X_2$ such that for any object $Y \in \operatorname{ob} \mathcal{C}$ and morphisms $f_i : Y \to X_i$ there exists a unique morphism $Y \to X_1 \times X_2$ such that the diagram

$$\begin{array}{ccc} Y & \dashrightarrow & X_1 \times X_2 \\ & {\scriptstyle f_i}\searrow \quad \swarrow{\scriptstyle \lambda_j} & \\ & X_i & \end{array}$$

commutes. In other words, $X_1 \times X_2$ is a **product**. Dually, a colimit for D is referred to as a **coproduct**.

Example 4.14. Let J be the category represented by $\cdot \rightrightarrows \cdot$ Then a diagram $D : J \to \mathcal{C}$ is a parallel pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$. Then a cone over D is an object E and morphisms $E \xrightarrow{h} A$ and $E \xrightarrow{k} B$ such that $k = fh = gh$. This is equivalent to an object E and a morphism $E \xrightarrow{h}$ such that $fh = gh$. A limit for D is thus such a cone such that for any other cone consisting of an object F and a morphism k with $fk = gk$, k factors through f uniquely:

$$\begin{array}{ccccc} E & \xrightarrow{h} & A & \overset{f}{\underset{g}{\rightrightarrows}} & B \\ \uparrow & \nearrow_{k} & & & \\ F & & & & \end{array}$$

In other words, a limit for D is an equaliser for the maps f and g. Dually, a colimit for D is a coequaliser.

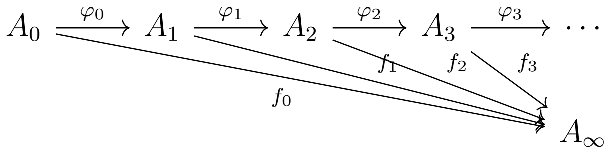
Example 4.15. Let J be the category represented by $\begin{array}{ccc} & & \cdot \\ & & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$ Then a diagram $D : J \to \mathcal{C}$ is of the form $\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$. So a cone consists of an object D and morphisms

$D \xrightarrow{h} A, D \xrightarrow{l} C, D \xrightarrow{k} B$ such that $fh = l = gk$. This is equivalent to a pair of morphisms $D \xrightarrow{h} A$ and $D \xrightarrow{k} B$ completing the diagram to a commutative square. A terminal such cone is called a **pullback** (or **fibred product**). Dually, a colimit for D is called a **pushout**.

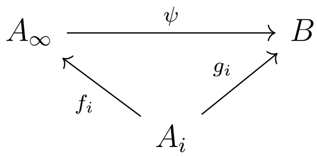
Example 4.16. Let $J = \mathbb{N}$ be the category whose objects are natural numbers and morphisms $n_1 \to n_2$ are order relations $n_1 < n_2$. Then a diagram of shape J is of the form

$$A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

A cocone under D is of the form



An initial such cocone is one such that for any other object B and morphisms $g_i : A_i \to B$ there is a unique morphism $\psi : A_\infty \to B$ such that the diagram



commutes. Dually, a limit for D is called an **inverse limit**.

Theorem 4.17 ($\star$)**.** *Let $\mathcal{C}$ be a category.*

1. *If $\mathcal{C}$ has equalisers and all finite (respectively small) products then $\mathcal{C}$ has all finite (respectively small) limits.*
2. *If $\mathcal{C}$ has pullbacks and a terminal object then $\mathcal{C}$ has all finite limits.*

Proof.

Part 1: Given a diagram $D : J \to \mathcal{C}$ with J finite (or small). We first form the products

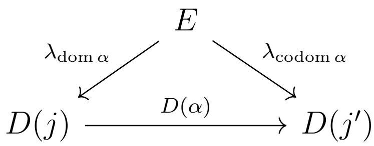
$$P = \prod_{j \in \operatorname{ob} J} D(j), \quad Q = \prod_{\alpha \in \operatorname{mor} J} D(\operatorname{codom} \alpha)$$

Let π_j^P and π_α^Q be the projection morphisms for the products P and Q respectively. Define a parallel pair $P \underset{g}{\overset{f}{\rightrightarrows}} Q$ by

$$\begin{aligned} \pi_\alpha^Q f &= \pi_{\operatorname{codom} \alpha}^P : P \to D(\operatorname{codom} \alpha) \\ \pi_\alpha^Q g &= D(\alpha) \pi_{\operatorname{dom} \alpha}^P : P \to D(\operatorname{dom} \alpha) \to D(\operatorname{codom} \alpha) \end{aligned}$$

Now form the equaliser $e : E \to P$ of f and g. Set $\lambda_j = \pi_j^P e : E \to D(j)$. We claim that $(E, (\lambda_j))$ is a limit cone for D.

We first show that $(E, (\lambda_j))$ is a cone over D. We need to show that for all $\alpha : j \to j'$ the diagram



commutes. By definition of f and g we have

$$D(\alpha)(\lambda_{\operatorname{dom}\alpha}) = D(\alpha)(\pi^P_{\operatorname{dom}\alpha} e) = \pi^Q_\alpha(ge) = \pi^Q_\alpha(fe) = \pi^P_{\operatorname{codom}\alpha}(e) = \lambda_{\operatorname{codom}\alpha}$$

as desired. Finally, we must show that $(E, (\lambda_j))$ is terminal in $\mathbf{Cone}(D)$. To this end, let $(C, (\mu_j))$ be another cone over D. By the universal property of products there exits a unique morphism $\mu : C \to P$ through which the μ_j factor: $\mu_j = \pi^P_j \mu$. Since the μ_j are legs of a cone, it then follows that for all $\alpha : j \to j'$ we have $\pi^Q_\alpha f\mu = \pi^Q_\alpha g\mu$. This implies that $f\mu = g\mu$ and so μ equalisers f and g. By the universal property of equalisers, μ factors uniquely through e as $\mu = e\nu$ for some $\nu : C \to E$. Such a ν is the desired unique morphism of cones $(C, (\mu_j)) \to (E, (\lambda_j))$.

Part 2: By Part 1, it suffices to show that if $\mathcal{C}$ has pullbacks and a terminal object then $\mathcal{C}$ has all finite products and equalisers. To this end, let 1 be a terminal object of $\mathcal{C}$ and $X_1, X_2 \in \operatorname{ob}\mathcal{C}$. Consider the diagram

$$\begin{array}{ccc} & & X_1 \\ & & \downarrow \\ X_2 & \longrightarrow & 1 \end{array}$$

A pullback of this diagram completes it to a commutative square

$$\begin{array}{ccc} C & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow \\ X_2 & \longrightarrow & 1 \end{array}$$

such that for any object D and morphism $g_i : D \to X_i$, there exists a unique morphism f such that $g_i = f_i f$. This is exactly what it means for C to be the product of A and B and so $\mathcal{C}$ has binary products. We can iterate this process to obtain all finite products.

Now fix a parallel pair $f, g : A \to B$ and consider the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle (1_A, f)} \\ A & \xrightarrow{(1_A, g)} & A \times B \end{array}$$

A pullback of this diagram completes it to a commutative square

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle (1_A, f)} \\ A & \xrightarrow{(1_A, g)} & A \times B \end{array}$$

This implies that $1_A h = 1_A k$ and $fh = gk$ whence $fh = gh$. Since this must be universal among such cones, P along with its legs must form an equaliser for f and g. $\square$

Definition 4.18 ($\star$)**.** Let $\mathcal{C}, \mathcal{D}$ and J be categories, $F : \mathcal{C} \to \mathcal{D}$ a functor and $D : J \to \mathcal{C}$ a diagram.

1. We say that F **preserves limits of shape J** if given a limit cone $(L, (\lambda_j))$ for D the cone $(FL, (F\lambda_j))$ is a limit for $FD : J \to \mathcal{D}$.

2. We say that F **reflects limits of shape J** if given a cone $(L, (\lambda_j))$ over D such that $(FL, (F\lambda_j))$ is a limit for FD then $(L, (\lambda_j))$ is a limit for D.

3. We say that F **creates limits of shape J** if given a limit cone $(M,(\mu_j))$ for FD there exists a cone $(L,(\lambda_j))$ over D whose image is isomorphic to $(M,(\mu_j))$ under F and any such cone is a limit for D.

Remark. If D has limits of shape J and $F:\mathcal{C}\to\mathcal{D}$ creates them then $\mathcal{C}$ also has them and F both preserves and reflects them.

In Theorem 4.17, we can replace "$\mathcal{C}$ has" with either "$\mathcal{C}$ has and $F:\mathcal{C}\to\mathcal{D}$ preserves" or "$\mathcal{D}$ has and $F:\mathcal{C}\to\mathcal{D}$ creates".

Example 4.19. Consider the forgetful functor $U:\mathbf{Grp}\to\mathbf{Set}$. Then U creates all small limits. Indeed, U clearly creates all small products and equalisers. However, U does not preserve colimits since it does not preserve the initial object. Indeed, the trivial group is initial in **Grp** but the singleton set is not an initial object in **Set**.

Example 4.20. The forgetful functor $U:\mathbf{Top}\to\mathbf{Set}$ clearly preserves all limits and colimits but doesn't reflect them. Indeed, let J be the discrete category of two objects and $D:J\to\mathbf{Top}$ a diagram of shape J. Suppose that $(L,(\lambda_j))$ is a cone over D such that $(UL,(U\lambda_j))$ is a limit for UD. Then, clearly, UL is the product in **Set**. However, L is not necessarily the product in **Top** as there are multiple topologies on $X\times Y$ that make the projections continuous but are not the product topology (such as the box topology).

Example 4.21. The inclusion functor $I:\mathbf{AbGrp}\to\mathbf{Grp}$ reflects coproducts but doesn't preserve them. Indeed, the coproduct in **AbGrp** is $A\oplus B$ and in **Grp** it is the free product $A\star B$ which is never abelian unless A or B is the trivial group.

Proposition 4.22 ($\star$)**.** *Let $\mathcal{C},\mathcal{D}$ and J be categories. If $\mathcal{D}$ has limits of shape J then $[\mathcal{C},\mathcal{D}]$ has limits of shape J and the forgetful functor $U:[\mathcal{C},\mathcal{D}]\to\mathcal{D}^{\mathrm{ob}\,\mathcal{C}}$ creates them.*

Proof. Suppose we are given a diagram $D:J\to[\mathcal{C},\mathcal{D}]$. Then D is the same thing as a functor $J\times\mathcal{C}\to\mathcal{D}$. For all $A\in\mathrm{ob}\,\mathcal{C}$, Let $(LA,(\lambda_{j,A}))$ be a limit for the diagram $D(\cdot,A):J\to\mathcal{D}$. Given $f:A\to B$, we claim that the composite $LA\xrightarrow{\lambda_j}D(j,A)\xrightarrow{D(j,f)}D(j,B)$ is a cone over $D(\cdot,B)$. We need to show that for all $\alpha:j\to j'$ the diagram

$$\begin{array}{ccc} & LA & \\ {\scriptstyle D(j,f)\circ\lambda_j}\swarrow & & \searrow{\scriptstyle D(j',f)\circ\lambda_{j'}} \\ D(j,B) & \xrightarrow{D(\alpha,B)} & D(j',B) \end{array}$$

commutes. In other words, we need to show that $D(j',f)\circ\lambda_{j'}=D(\alpha,B)\circ D(j,f)\circ\lambda_j$. Since $(LA,(\lambda_{j,A}))$ is a cone, this is equivalent to showing that $D(j,f)\circ D(\alpha,A)\circ\lambda_j=D(\alpha,B)\circ D(j,f)\circ\lambda_j$. Note that the square

$$\begin{array}{ccc} (j,A) & \xrightarrow{(1_j,f)} & (j,B) \\ \downarrow{\scriptstyle(\alpha,A)} & & \downarrow{\scriptstyle(\alpha,B)} \\ (j',A) & \xrightarrow{(1_{j'},f)} & (j',B) \end{array}$$

commutes and so the induced square

$$\begin{array}{ccc} D(j,A) & \xrightarrow{D(1_j,f)} & D(j,B) \\ \downarrow{\scriptstyle D(\alpha,A)} & & \downarrow{\scriptstyle D(\alpha,B)} \\ D(j',A) & \xrightarrow{(1_{j'},f)} & (j',B) \end{array}$$

commutes whence the claim is proven. It then follows that, given a limit $(LB, (\lambda_{j,B}))$ over the diagram $D(\cdot, B)$, there is a unique morphism $Lf : LA \to LB$ such that diagram

$$\begin{array}{ccc} LA & \xrightarrow{Lf} & LB \\ \downarrow{\scriptstyle \lambda_{j,A}} & & \downarrow{\scriptstyle \lambda_{j,B}} \\ D(j,A) & \xrightarrow{D(j,f)} & D(j,B) \end{array}$$

commutes. We claim that $L : \mathcal{C} \to \mathcal{D}$ is a functor. To this end, let $f : A \to B$ and $g : B \to C$. We need to show that $L(gf) = (Lg)(Lf)$. We have that

$$\lambda_{j,C} \circ Lg \circ Lf = D(j,g) \circ D(j,f) \circ \lambda_{j,A} = D(j,gf)\lambda_{j,A} = \lambda_{j,C} \circ L(gf)$$

So $L(gf)$ and $Lg \circ Lf$ are both factorisations of the cone $(LA, (LA \to D(j,C)))$ through the limit LC and so they must be equal since LC is terminal. Hence L is a functor whence $(L, (\lambda_{j,\cdot}))$ is a cone over the diagram $D : J \to [\mathcal{C}, \mathcal{D}]$. We now claim that this cone is a limit for D. Let $(M, (\mu_{j,\cdot} : M \to D(j, \cdot)))$ be any other cone over D. Then $(MA, \mu_{j,A})$ is a cone over $D(\cdot, A)$ so there exists a unique $\nu_A : MA \to LA$ such that $\lambda_{j,A}\nu_A = \mu_{j,A}$ for all j. We claim that ν is a natural transformation and so $\nu \in \mathrm{mor}[\mathcal{C}, \mathcal{D}]$. We need to show that for all $f : A \to B$, the diagram

$$\begin{array}{ccc} MA & \xrightarrow{Mf} & MB \\ \downarrow{\scriptstyle \nu_A} & & \downarrow{\scriptstyle \nu_B} \\ LA & \xrightarrow{Lf} & LB \end{array}$$

commutes. But this is immediate from the uniqueness of μ_A, μ'_B, Mf and Lf. Hence ν is the unique factorisation of the $\mu_{j,\cdot}$ through $\lambda_{j,\cdot}$ and so $(L, (\lambda_{j,\cdot}))$ is a limit cone for D as desired. $\square$

Lemma 4.23 ($\star$). *Let $\mathcal{C}$ be a category and $f : A \to B$ a morphism in $\mathcal{C}$. Then f is a monomorphism if and only if the commutative square*

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback square.

Proof. Suppose that f is a monomorphism. Let

$$\begin{array}{ccc} C & \xrightarrow{g} & A \\ \downarrow{\scriptstyle h} & & \\ A & & \end{array}$$

be another cone. We need to show that there exists a unique morphism $z : C \to A$ such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{z} & A \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle 1_A} & \\ & A & \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{z} & A \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle 1_A} & \\ & A & \end{array}$$

commute. Since the cone with apex C completes the diagram to a commutative square such that $fg = fh$ and f is monic, we have that $g = h$. Hence g is the unique such morphism.

Conversely, suppose that the commutative square is a pullback square. Let

$$\begin{array}{ccc} C & \xrightarrow{g} & A \\ \downarrow{\scriptstyle h} & & \\ A & & \end{array}$$

be another cone. Then we have $fg = fh$ and there exists a unique morphism such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{z} & A \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle 1_A} & \\ & A & \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{z} & A \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle 1_A} & \\ & A & \end{array}$$

commute. Then $g = 1_A z = h$ and so f is monic. □

Proposition 4.24 ($\star$)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and suppose that $\mathcal{D}$ has pullbacks. Then $\alpha \in \operatorname{mor}[\mathcal{C}, \mathcal{D}]$ is monic if and only if α_A is monic for all $A \in \operatorname{ob}\mathcal{C}$.*

Proof. Suppose that $\alpha \in \operatorname{mor}[\mathcal{C}, \mathcal{D}]$. Then $\alpha : F \to G$ is monic if and only if the commutative square

$$\begin{array}{ccc} F & \xrightarrow{1_F} & F \\ \downarrow{\scriptstyle 1_F} & & \downarrow{\scriptstyle \alpha} \\ F & \xrightarrow{\alpha} & G \end{array}$$

is a pullback square. By 4.22, the forgetful functor $U : [\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\operatorname{ob}\mathcal{C}}$ creates all pullbacks. Hence the above diagram is a pullback square if and only if

$$\begin{array}{ccc} FA & \xrightarrow{1_{FA}} & FA \\ \downarrow{\scriptstyle 1_{FA}} & & \downarrow{\scriptstyle \alpha_A} \\ FA & \xrightarrow{\alpha_A} & GA \end{array}$$

is a pullback square for all $A \in \operatorname{ob}\mathcal{C}$. This is equivalent to α_A being a monomorphism for all $A \in \operatorname{ob}\mathcal{C}$. □

Theorem 4.25 ($\star$)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $G : \mathcal{D} \to \mathcal{C}$ a functor. If G has a left adjoint then G preserves all limits which exist in $\mathcal{D}$.*

Proof. Suppose that $F \dashv G$ is an adjunction and J is a category. Define a functor $F^J : [J, \mathcal{C}] \to [J, \mathcal{D}]$ by $F^J(D) = FD$ on objects and $F^J(\alpha) = F\alpha$ on morphisms. We similarly define $G^J : [J, \mathcal{D}] \to [J, \mathcal{C}]$. We first claim that $F^J \dashv G^J$. Let $\eta : 1_\mathcal{C} \to GF$ and $\eta : FG \to 1_\mathcal{D}$ be the unit and counit respectively of the adjunction $F \dashv G$. We need to Simi exhibit natural transformations $\eta^J : 1_{[J,\mathcal{C}]} \to G^J F^J$ and $\varepsilon^J : F^J G^J \to 1_{[J,\mathcal{D}]}$ that satisfy the triangular identities. Given $S : J \to \mathcal{C}$, define $\eta^J_S = \eta \circ 1_S : S \to GFS$. We first verify that this is a natural transformation. We need to show that for all $\alpha : S \to S'$, the diagram

$$\begin{array}{ccc} S & \xrightarrow{\eta^J_S} & GFS \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle GF\alpha} \\ S' & \xrightarrow{\eta^J_{S'}} & GFS' \end{array}$$

But this is clear from the commutativity of the diagram

$$\begin{array}{ccc} SA & \xrightarrow{\eta^J_{SA}} & GFSA \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle GF\alpha_A} \\ S'A & \xrightarrow{\eta^J_{S'A}} & GFS'A \end{array}$$

for all $A \in \operatorname{ob} J$. Similarly, given $S : J \to \mathcal{D}$, we define $\varepsilon^J_S = \varepsilon \circ 1_S : FGS \to S$ which is natural in S. We now show that these natural transformations satsify the triangular identities. We are required to show that the diagram

$$\begin{array}{ccc} F^J & \xrightarrow{F^J\eta^J} & F^JG^JF^J \\ & {\scriptstyle 1_{F^J}} \searrow & \downarrow{\scriptstyle \varepsilon^J_{F^J}} \\ & & F^J \end{array}$$

commutes. This is equivalent to showing that for all functors $S : J \to \mathcal{C}$ and objects $A \in \operatorname{ob} J$ the diagram

$$\begin{array}{ccc} FSA & \xrightarrow{F\eta_{SA}} & FGFSA \\ & {\scriptstyle 1_{FSA}} \searrow & \downarrow{\scriptstyle \varepsilon_{FSA}} \\ & & FSA \end{array}$$

commutes. But this follows immediately from the fact that η satisfies the triangular identities. Hence $F^J \dashv G^J$. Now suppose that $\mathcal{C}$ and $\mathcal{D}$ have limits of shape J. Let $\triangle^{\mathcal{C}} : C \to [J, C]$ which takes an object C to the constant diagram of shape J on C. Similarly, define $\triangle^{\mathcal{D}}$. By definition, $\mathcal{C}$ and $\mathcal{D}$ have limits of shape J if $\triangle^{\mathcal{C}}$ and $\triangle^{\mathcal{D}}$ have right adjoints. Suppose that $\triangle^{\mathcal{C}} \dashv \varprojlim^{\mathcal{C}}_J$ and $\triangle^{\mathcal{D}} \dashv \varprojlim^{\mathcal{D}}_J$. We can then form the composite adjunctions $\triangle^{\mathcal{D}} \circ F \dashv G \circ \varprojlim^{\mathcal{D}}_J$ and $F^J \circ \triangle^{\mathcal{C}} \dashv \varprojlim^{\mathcal{C}}_J \circ G^J$. We now claim that $\triangle^{\mathcal{D}} \circ F = F^J \circ \triangle^{\mathcal{C}}$. Indeed, let $C \in \operatorname{ob} \mathcal{C}$. Then $F^J \circ \triangle^{\mathcal{C}}(C) = F\triangle^{\mathcal{C}}(C)$. But this is the same as the constant diagram of shape J over FC which is equal to $\triangle^{\mathcal{D}}(FC)$. We thus see that $G \circ \varprojlim^{\mathcal{D}}_J$ and $\varprojlim^{\mathcal{C}}_J \circ G^J$ are right adjoint to the same functor and so they must be naturally isomorphic. In other words, for all functors $S : J \to \mathcal{D}$ we have

$$\varprojlim^{\mathcal{C}}_J (GJ) \cong G(\varprojlim^{\mathcal{D}}_J (J))$$

But this is exactly what it means for G to preserve limits of shape J. $\square$

Lemma 4.26. *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ has an initial object if and only if $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ has a limit.*

Proof. Let 0 be an initial object of $\mathcal{C}$. Then for each $A \in \operatorname{ob} \mathcal{C}$, there is a unique morphism $\lambda_A : 0 \to A$. We claim that $(0, (\lambda_A))$ form a cone over $1_{\mathcal{C}}$. We need to show that for all $f : A \to B$ the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & {\scriptstyle \lambda_A} \swarrow & & \searrow {\scriptstyle \lambda_B} & \\ A & & \xrightarrow{\quad f \quad} & & B \end{array}$$

commutes. But this is exactly what it means for 0 to be initial. We now claim that $(0, (\lambda_A))$ is terminal in $\mathbf{Cone}(1_{\mathcal{C}})$. Indeed, suppose that $(L, (\mu_A))$ is another cone over $1_{\mathcal{C}}$. Then clearly, $\mu_0 : L \to 0$ is the unique morphism such that $\mu_A = \lambda_A \mu_0$ and so $(0, (\lambda_A))$ is terminal.

Conversely, suppose that $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ has a limit, say $(L, (\lambda_A))$. We claim that L is initial in $\mathcal{C}$. Suppose that $f : L \to A$ for some $A \in \operatorname{ob} \mathcal{C}$. By definition, we have that the diagram

$$\begin{array}{ccc} & L & \\ {\scriptstyle \lambda_L} \swarrow & & \searrow {\scriptstyle \lambda_A} \\ L & \xrightarrow{\quad f \quad} & A \end{array}$$

commutes. In particular, if $f = \lambda_A$ then for all $A \in \operatorname{ob} \mathcal{C}$ we have $\lambda_A \lambda_L = \lambda_A$. Since $(L, (\lambda_A))$ is terminal, it follows that $\lambda_L = 1_L$ whence $f = \lambda_A$ and so L is initial in $\mathcal{C}$. □

Lemma 4.27. *Let $\mathcal{C}, \mathcal{D}$ and J be categories. Suppose that $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves limits of shape J. Then for all $A \in \operatorname{ob} \mathcal{C}$, $(A \downarrow G)$ has limits of shape J and the forgetful functor $U : (A \downarrow G) \to \mathcal{D}$ creates them.*

Proof. Fix $A \in \operatorname{ob} \mathcal{C}$ and let $D : J \to (A \downarrow G)$ be a diagram of shape J. For each j, we can write $D(j)$ as $(UD(j), f_j : A \to GUD(j))$. We first claim that $(A, (f_j))$ are a cone over the diagram $GUD : J \to \mathcal{C}$. Suppose that we are given $\alpha : j \to j'$. We need to show that the diagram

$$\begin{array}{ccc} & A & \\ {\scriptstyle f_j} \swarrow & & \searrow {\scriptstyle f_{j'}} \\ GUD(j) & \xrightarrow{\quad GUD(\alpha) \quad} & GUD(j') \end{array}$$

commutes. But this follows immediately from the definition of a morphism in $(A \downarrow G)$. Since $\mathcal{D}$ has limits of shape J, the diagram UD has a limit, say $(L, (\lambda_j))$. Since G preserves limits of shape J, $(GL, (G\lambda_j))$ is a limit for the diagram GUD. Hence there must exist a unique morphism $h : A \to GL$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad h \quad} & GL \\ {\scriptstyle f_j} \searrow & & \swarrow {\scriptstyle G\lambda_j} \\ & GUD(j) & \end{array}$$

commutes for all j. Hence the λ_j are morphisms $(L, h) \xrightarrow{\lambda_j} (UD(j), f_j)$ in $(A \downarrow G)$. We claim that $((L, h), (\lambda_j))$ form a cone over D. To this end, given $\alpha : j \to j'$, we must show that the diagram

$$\begin{array}{ccc} & (L, h) & \\ {\scriptstyle \lambda_j} \swarrow & & \searrow {\scriptstyle \lambda_{j'}} \\ (UD(j), f_j) & \xrightarrow{\quad (UD(\alpha), f_\alpha) \quad} & (UD(j'), f_{j'}) \end{array}$$

commutes. However, this is immediate as U is faithful and $(L, (\lambda_j))$ form a cone over UD. We now claim that $((L, h), (\lambda_j))$ is a limit cone for D. To this end, let $((B, f), (\mu_j))$ be any other cone over D. Then $(B, (\mu_j))$ is a cone over UD. Since $(L, (\lambda_j))$ is a limit for UD, there exists a unique $k : B \to L$ such that $\lambda_j k = \mu_j$ for all j. To show that k is a morphism in $(A \downarrow G)$, we need to show that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ g \ } & GB \\ & {\scriptstyle h} \searrow & \downarrow {\scriptstyle Gk} \\ & & GL \end{array}$$

commutes. But $(GL, (G\lambda_j))$ is a limit for the diagram GUD and so the two ways round the diagram must be equal. Hence there is a unique morphism $((B, f), (\mu_j)) \xrightarrow{k} ((L, h), (\lambda_j))$ such that $\lambda_j k = \mu_j$ whence $((L, h), (\lambda_j))$ is terminal. □

Theorem 4.28 (Primeval Adjoint Functor Theorem)**.** *Let $\mathcal{C}, \mathcal{D}$ be categories. If $\mathcal{D}$ has all limits and $G : \mathcal{D} \to \mathcal{C}$ preserves them then G has a left adjoint.*

Proof. By Lemma 4.27, $(A \downarrow G)$ has all limits for all $A \in \operatorname{ob}\mathcal{C}$. By Lemma 4.26, $(A \downarrow G)$ has an initial object for each $A \in \operatorname{ob}\mathcal{C}$. By Theorem 3.12, G therefore has a left adjoint. □

Remark. This theorem is, however, too strong to be of use in an arbitrary category. Indeed, if a category $\mathcal{D}$ has all limits as big as itself then it is a preorder[2]. To see this we may assume, without loss of generality, that $\mathcal{D}$ is small so that $|\operatorname{ob}\mathcal{D}| = |\operatorname{mor}\mathcal{D}| = \kappa$ for some cardinal number κ. Suppose that $\mathcal{D}$ has products of size κ. Given a distinct parallel pair $f, g : A \to B$ we can form the product $P = \prod_{h \in \operatorname{mor}\mathcal{D}} B$. Then

$$|\mathcal{D}(A, P)| = |P|^{|A|} = \left(\left| \prod_{h \in \operatorname{mor}\mathcal{D}} B \right| \right)^{|A|} = (|B|^{\kappa})^{|A|} = (|B|^{|A|})^{\kappa} = (|\mathcal{D}(A, B)|)^{\kappa} \geq 2^{\kappa}$$

But this contradicts the fact that $|\operatorname{mor}\mathcal{D}| = \kappa$. Hence $f = g$ and $\mathcal{D}$ is a preorder.

Definition 4.29. Let $\mathcal{D}$ be a category. We say that $\mathcal{D}$ is **complete** if it has all small limits.

Theorem 4.30 (General Adjoint Functor Theorem)**.** *Let $\mathcal{D}$ be a locally small complete category. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if it preserves all small limits and sastifies the* ***solution-set condition****: for all $A \in \operatorname{ob}\mathcal{C}$, there exists a set $\{ (B_i, A \xrightarrow{f_i} GB_i) \mid i \in I \}$ of objects of $(A \downarrow G)$ such that any $A \xrightarrow{h} GC$ factors as $A \xrightarrow{f_i} GB_i \xrightarrow{Gg} Gc$ for some $i \in I$ and $g : B \to C$.*

Proof. First suppose that G has a left adjoint F. Then by Theorem 4.25, G preserves all small limits. Let $\eta : 1_{\mathcal{C}} \to GF$ be the unit of the adjunction $F \dashv G$. Given $A \in \operatorname{ob}\mathcal{C}$, we claim that $\{ (FA, A \xrightarrow{\eta_A} GFA) \}$ is a solution set for A. But this is clear from Corollary 3.12 which asserts that (FA, η_A) is initial in $(A \downarrow G)$.

Conversely, suppose that $G : \mathcal{D} \to \mathcal{C}$ preserves all small limits and for each $A \in \operatorname{ob}\mathcal{C}$, there is a set of objects of $(A \downarrow G)$ satisfying the solution set condition - we shall refer to such a set as **weakly initial**. By Lemma 4.27, $(A \downarrow G)$ is complete and clearly inherits the local smallness of $\mathcal{D}$. If we can show that $(A \downarrow G)$ has an initial object then Corollary 3.12 would show that G has a left adjoint.

To this end, let $\mathcal{A}$ be a complete locally small category with a weakly inital set of objects $\{ S_i \mid i \in I \}$. We need to show that $\mathcal{A}$ has an initial object. First form the product $P = \prod_{i \in I} S_i$. Then P is also weakly initial. Indeed, the product comes equipped with projections $\pi_i : P \to S_i$ for each i. Given $A \in \operatorname{ob}\mathcal{A}$, there is a morphism $S_i \xrightarrow{f_i} A$ and so $f_i \pi_i$ is a morphism $P \to A$. Now consider the diagram

$$P \rightrightarrows P$$

whose edges are all the endomorphisms of P in $\mathcal{A}$. Form the limit $a : I \to P$ of this diagram. Note that I (as a singleton set) is weakly initial since P is. We claim that, in fact, I is initial. Suppose $f, g : I \to A$ are a distinct parallel pair for some $A \in \operatorname{ob}\mathcal{A}$. Let

[2] A preorder is a category with at most one morphism between any two given objects

$b : E \to I$ be an equaliser for them. Since P is weakly initial, there exists $c : P \to E$. Hence $P \xrightarrow{c} E \xrightarrow{b} I \xrightarrow{a} P$ is an edge of the diagram. Since 1_P is also an edge of the diagram and I is a limit, we must have that $abca = 1_P a = a$. Note that I is a generalised equaliser and so a is monic. We then have that $bca = 1_I$. In particular, b is split epic and since b is also regular monic, it follows that b is an isomorphism. Hence $f = g$ and so I is initial in $\mathcal{A}$. □

Example 4.31. Consider the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$. Suppose that we did not know how to construct free groups. We could use the General Adjoint Functor Theorem to construct a left adjoint for U. Indeed, $\mathbf{Grp}$ has and U preserves all small limits and $\mathbf{Grp}$ is locally small. Given a set A and a function $f : A \to UG$, f factors uniquely as $A \to UG' \to UG$ where G' is the subgroup of G generated by the set $\{ f(a) \mid a \in A \}$. Clearly $|G'| \leq \max\{\aleph_0, |A|\}$ and so the G' induce a solution set.

Definition 4.32. Let $\mathcal{C}$ be a category and $A \in \operatorname{ob}\mathcal{C}$ an object. By a **subobject** A' of A, we mean a monomorphism $A' \rightarrowtail A$. We denote the full subcategory of $(1_F \downarrow A)$ whose objects are the subobjects of A by $\mathbf{Sub}_{\mathcal{C}}(A)$.

Definition 4.33 ($\star$)**.** Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is **well-powered** if $\mathbf{Sub}_{\mathcal{C}}(A)$ is equivalent to a small category for all $A \in \operatorname{ob}\mathcal{C}$. In other words, up to isomorphism, each object of $\mathcal{C}$ has only a set of subobjects.

Lemma 4.34. *Let $\mathcal{C}$ be a category and suppose we are given a pullback square*

$$\begin{array}{ccc} P & \xrightarrow{k} & A \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

of objects and morphisms in $\mathcal{C}$ with f monic. Then h is monic.

Proof. Suppose we are given a parallel pair $x, y : D \to P$ such that $hx = hy$. Then

$$fkx = ghx = ghy = fky$$

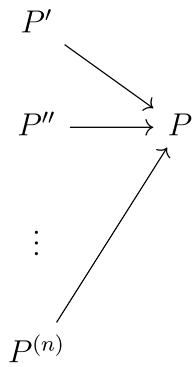
Since f is a monomorphism, we have that $kx = ky$. From this it follows that $x = y$ since they are both factorisations of the cone of apex D through the pullback. □

Theorem 4.35 (Special Adjoint Functor Theorem)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be locally small categories and assume that $\mathcal{D}$ is complete, well-powered and has a coseperating set. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if G preserves all small limits.*

Proof. The forward direction is simply Theorem 4.25.

Conversely, suppose that G preserves all small limits. We first claim that $(A \downarrow G)$ has all the properties that we have assumed for $\mathcal{D}$. By Lemma 4.27, $(A \downarrow G)$ has all small limits and inherits the local smallness of $\mathcal{D}$. $(A \downarrow G)$ is well powered since subobjects of (B, f) are in one to one correspondence with subobjects $B' \rightarrowtail B$ such that f factors through $GB' \rightarrowtail GB$. Finally, $(A \downarrow G)$ has a coseperating set. Indeed, if $\{ S_i \mid i \in I \}$ is a coseperating set for $\mathcal{D}$ then $\{ (S_i, f) \mid i \in I, f : A \to GS_i \}$ is a coseperating set for $(A \downarrow G)$. To see this, suppose that $g, g' : (B, f) \to (B', f')$ are two distinct morphisms. Then there exists $h : B' \to S_i$ for each i such that $hg \neq hg'$ and then h is a morphism $(B', f) \to (S_i, (Gh)f')$ in $(A \downarrow G)$. If we can show that $(A \downarrow G)$ has an initial object then we are done by Corollary 3.12.

To this end, suppose that $\mathcal{A}$ is complete, locally small, well-powered and has a coseperating set $\{ S_i \mid i \in I \}$. We first form the product $P = \prod_{i \in I} S_i$. Consider the diagram



whose edges are a (possibly infinite) representative set of subobjects of P. Let I be a limit for this diagram. By Lemma 4.34, each leg $I \to P^{\cdot}$ is monic and so there exists a monomorphism $I \rightarrowtail P$. This clearly implies that I is the least subobject of P since any other monomorphism into the P^{-} would have to factor through I. We claim that I is initial.

We first show the uniqueness property. Suppose that $f, g : I \to A$ is a parallel pair. Let $e : E \to I$ be the equaliser of f and g. Then since e is monic, E is a subobject of I. But I has no proper subobjects and so e must be an isomorphism and $f = g$.

Now fix $A \in \operatorname{ob} \mathcal{A}$, we need to show that there exists a morphism $I \to A$. We form the product

$$Q = \prod_{i, f : A \to S_i} S_i$$

The morphism $h : A \to Q$ defined by $\pi_{i,f} h = f$ is clearly monic since the S_i form a coseperating set. We also have $k : P \to Q$ defined by $\pi_{i,f} k = \pi_i$. Forming the pullback of h and k we have

$$\begin{array}{ccc} B & \xrightarrow{m} & A \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle h} \\ P & \xrightarrow{k} & Q \end{array}$$

By Lemma 4.34, l is monic and hence isomorphic to an edge of the diagram defining I. Hence $I \rightarrowtail P$ must factor through l. In particular, there exists a morphism $I \to B$ which, composed with m, gives us a morphism $I \to A$. □

Example 4.36. The Stone-Čech compactification is a special case of the Special Adjoint Functor Theorem. Indeed, consider the inclusion functor $F : \mathbf{KHaus} \to \mathbf{Top}$. **KHaus** has and F preserves all small products by Tychonoff's Theorem. F also preserves equalisers. Indeed, given a parallel pair $f, g : X \rightrightarrows Y$ where Y is Hausdorff, the equaliser E is a closed subspace of X. To see this, let $(f, g) : X \to Y \times Y$ denote the continuous map that sends x to $(f(x), g(x))$. Then $E = (f, g)^{-1}(\triangle_Y)$ where $\triangle_Y$ is the diagonal. Since in any Hausdorff space the diagonal is closed, it follows that E is closed in X.

KHaus is also clearly well-powered as subobjects of X correspond up to isomorphism to closed subspaces of X. Finally, **KHaus** has a coseperator $[0, 1]$ by Urysohn's Lemma. Therefore, the Special Adjoint Functor Theorem implies that F has a left adjoint.

5 Monads

Definition 5.1 ($\star$)**.** Let $\mathcal{C}$ be a category. A **monad** $\mathbb{T}$ on $\mathcal{C}$ is a triple (T, η, μ) where $T : \mathcal{C} \to \mathcal{C}$ is a functor and $\eta : 1_{\mathcal{C}} \to T$ and $\mu : TT \to T$ are natural transformations such that the diagrams

$$
\begin{array}{ccccc}
T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta T} & T \\
& \searrow^{1_T} \; ① & \downarrow \mu & ② \; \swarrow^{1_T} & \\
& & T & &
\end{array}
\qquad
\begin{array}{ccc}
TTT & \xrightarrow{\mu T} & TT \\
\downarrow T\mu & ③ & \downarrow \mu \\
TT & \xrightarrow{\mu} & T
\end{array}
$$

Example 5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $F \dashv G$ is an adjunction of $\mathcal{C}$ and $\mathcal{D}$ with unit η and counit ε. Then $\mathbb{T} = (GF, \eta, G\varepsilon_F)$ is a monad on $\mathcal{C}$.

Example 5.3. Let M be a monoid and consider the functor $M \times (\cdot) : \mathbf{Set} \to \mathbf{Set}$ which takes a set A and sends it to the cartesian product $M \times A$. Then $M \times (\cdot)$ has a monad structure given by the natural transformations

$$
\eta_A(a) = (1_M, a)
$$
$$
\mu_A(m, m', a) = (mm', a)
$$

The fact that this is a monad follows directly from the axioms of a monoid.

Definition 5.4. Let $\mathcal{C}$ be a category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. By a **$\mathbb{T}$-algebra**, we mean a pair (A, α) where $A \in \operatorname{ob}\mathcal{C}$ is an object and $\alpha : TA \to A$ is a morphism satisfying the commutative diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & TA \\
& \searrow^{1_A} \; ④ & \downarrow \alpha \\
& & A
\end{array}
\qquad
\begin{array}{ccc}
TTA & \xrightarrow{T\alpha} & TA \\
\downarrow \mu_A & ⑤ & \downarrow \alpha \\
TA & \xrightarrow{\alpha} & A
\end{array}
$$

A **homomorphism** of $\mathbb{T}$-algebras $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ such that the diagram

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\downarrow \alpha & ⑥ & \downarrow \beta \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes. We write $\mathcal{C}^{\mathbb{T}}$ for the category of all $\mathbb{T}$-algebras together with their homomorphisms.

Lemma 5.5. *Let $\mathcal{C}$ be a category and $\mathbb{T}$ a monad on $\mathcal{C}$. Then the forgetful functor $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ has a left adjoint $F^{\mathbb{T}}$ and the adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ induces the monad $\mathbb{T}$.*

Proof. Suppose that $\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathcal{C}$. We shall define a functor $F^{\mathbb{T}} : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ as follows: On objects $A \in \operatorname{ob}\mathcal{C}$ we set $F^{\mathbb{T}}A = (TA, \mu_A)$. This is clearly an algebra by diagrams 2. and 3. On morphisms we set $F^{\mathbb{T}}(A \xrightarrow{f} B) = Tf$ which is a homomorphism by the naturality of μ. Note that $G^{\mathbb{T}}F^{\mathbb{T}} = T$ and we already have a natural transformation $\eta : 1_{\mathcal{C}} \to T$ from the definition of the monad. This gives us the unit of the claimed adjunction. We define the counit $\varepsilon : F^{\mathbb{T}}G^{\mathbb{T}} \to 1_{\mathcal{C}^{\mathbb{T}}}$ by $\varepsilon_{(A,\alpha)} = \alpha : TA \to A$ which is a morphism $\alpha : (TA, \mu) \to (A, \alpha)$ in $\mathcal{C}^{\mathbb{T}}$ by diagram 5. and this transformation is natural by diagram 6. The triangular identities follow immediately from diagrams 4. and 1. so $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ is an adjunction. □

Definition 5.6. Let $\mathcal{C}$ be a category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. We define the **Kleisli category**, denoted $\mathcal{C}_{\mathbb{T}}$ to be the one whose objects are $\operatorname{ob}\mathcal{C}$ and whose morphisms $A \dashrightarrow B$ are morphisms $A \to TB$ in $\mathcal{C}$. Given $A \in \operatorname{ob}\mathcal{C}_{\mathbb{T}}$, the identity morphism $A \overset{1_A}{\dashrightarrow} A$ is $A \xrightarrow{\eta_A} TA$. The composite $A \overset{f}{\dashrightarrow} B \overset{g}{\dashrightarrow} C$ is given by $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC$.

Remark. In order for this to actually be a category, we must check that composition with the identity preserves a morphism and that the composition of morphisms is associative. To this end, let $A \overset{f}{\dashrightarrow} B$ be a morphism in $\mathcal{C}_{\mathbb{T}}$. Then by the naturality of η and diagram 2 we have

$$\begin{aligned} A \overset{1_A}{\dashrightarrow} A \overset{f}{\dashrightarrow} B &= A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \\ &= A \xrightarrow{f} TB \xrightarrow{\eta_{TB}} TTB \xrightarrow{\mu_B} TB \\ &= A \xrightarrow{f} TB \end{aligned}$$

Furthermore, let $A \overset{f}{\dashrightarrow} B \overset{g}{\dashrightarrow} C \overset{h}{\dashrightarrow} D$ be a composition of morphisms in $\mathcal{C}_{\mathbb{T}}$. By diagram 3 and the naturality of μ, we have

$$\begin{aligned} (hg)f &= A \xrightarrow{f} TB \xrightarrow{T(hg)} TTD \xrightarrow{\mu_D} TD \\ &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{T\mu_D} TTD \xrightarrow{\mu_D} TD \\ &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{\mu_{TD}} TTD \xrightarrow{\mu_D} TD \\ &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \xrightarrow{Th} TTD \xrightarrow{\mu_D} TD \\ &= A \xrightarrow{gf} TC \xrightarrow{Th} TTD \xrightarrow{\mu_D} TD \\ &= h(gf) \end{aligned}$$

Proposition 5.7. *Let $\mathcal{C}$ be a category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. Then there exists an adjunction $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ in $\mathcal{C}_{\mathbb{T}}$ which induces $\mathbb{T}$.*

Proof. We define $F_{\mathbb{T}}$ on objects to be $F_{\mathbb{T}}A = A$ and on morphisms by $F_{\mathbb{T}}(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. Then $F_{\mathbb{T}}$ clearly preserves identities. To see that it is a functor, note that

$$\begin{aligned} (F_{\mathbb{T}}g)(F_{\mathbb{T}}f) &= A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} TTC \xrightarrow{\mu_C} TC \\ &= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \\ &= F_{\mathbb{T}}(gf) \end{aligned}$$

where we have used diagram 1 and the naturality of η.

Conversely, define $G_{\mathbb{T}}$ on objects by $G_{\mathbb{T}}A = TA$ and on morphisms by $G_{\mathbb{T}}(A \overset{f}{\dashrightarrow} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. By diagram 1, this preserves identities. To see that it is a functor, note that

$$\begin{aligned} (G_{\mathbb{T}}g)(G_{\mathbb{T}}f) &= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \\ &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{\mu_{TC}} TTC \xrightarrow{\mu_C} TC \\ &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{T\mu_C} TTC \xrightarrow{\mu_C} TC \\ &= G_{\mathbb{T}}(gf) \end{aligned}$$

It is obvious that $G_{\mathbb{T}}F_{\mathbb{T}}A = TA$ and

$$G_{\mathbb{T}}F_{\mathbb{T}}(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} TB = Tf$$

by diagram 1. It then follows that $G_{\mathbb{T}}F_{\mathbb{T}} = T$. We already have a unit for the adjunction $\eta : 1_{\mathbb{C}} \to G_{\mathbb{T}}F_{\mathbb{T}}$. We define the counit $F_{\mathbb{T}}G_{\mathbb{T}}A \xrightarrow{\varepsilon_A} A$ to be $TA \xrightarrow{1_{TA}} TA$. To verify the naturality of ε, consider the diagram

$$\begin{array}{ccc} F_{\mathbb{T}}G_{\mathbb{T}}A & \overset{F_{\mathbb{T}}G_{\mathbb{T}}f}{\dashrightarrow} & F_{\mathbb{T}}G_{\mathbb{T}}B \\ {\scriptstyle \varepsilon_A}\downarrow & & \downarrow{\scriptstyle \varepsilon_B} \\ A & \overset{f}{\dashrightarrow} & B \end{array}$$

We have that

$$\begin{aligned}
\varepsilon_B \circ F_{\mathbb{T}}G_{\mathbb{T}}f &= F_{\mathbb{T}}G_{\mathbb{T}}A \overset{F_{\mathbb{T}}G_{\mathbb{T}}f}{\dashrightarrow} F_{\mathbb{T}}G_{\mathbb{T}}B \overset{\varepsilon_B}{\dashrightarrow} B \\
&= F_{\mathbb{T}}G_{\mathbb{T}}A \xrightarrow{F_{\mathbb{T}}G_{\mathbb{T}}f} TF_{\mathbb{T}}G_{\mathbb{T}}B \xrightarrow{T\varepsilon_B} TTB \xrightarrow{\mu_B} TB \\
&= TA \xrightarrow{Tf} TTB \xrightarrow{1_{TTB}} TTB \xrightarrow{\mu_B} TB \\
&= TA \xrightarrow{1_{TA}} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \\
&= f \circ \varepsilon_A
\end{aligned}$$

and so ε is a natural transformation. We next check the triangular identities. By diagram 2, we have

$$\begin{aligned}
G_{\mathbb{T}}A \xrightarrow{\eta_{F_{\mathbb{T}}A}} G_{\mathbb{T}}F_{\mathbb{T}}G_{\mathbb{T}}A \xrightarrow{G\varepsilon_A} G_{\mathbb{T}}A &= TA \xrightarrow{\eta_A} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA \\
&= TA \xrightarrow{\eta_A} TTA \xrightarrow{\mu_A} TA \\
&= 1_{TA}
\end{aligned}$$

Dually we have

$$\begin{aligned}
F_{\mathbb{T}}A \overset{F_{\mathbb{T}}\eta_A}{\dashrightarrow} F_{\mathbb{T}}G_{\mathbb{T}}F_{\mathbb{T}}A \overset{\varepsilon_{F_{\mathbb{T}}A}}{\dashrightarrow} F_{\mathbb{T}}A &= F_{\mathbb{T}}A \xrightarrow{F\eta_A} TF_{\mathbb{T}}G_{\mathbb{T}}F_{\mathbb{T}}A \xrightarrow{T\varepsilon_{F_{\mathbb{T}}A}} TTFA \xrightarrow{\mu_{F_{\mathbb{T}}A}} F_{\mathbb{T}}A \\
&= A \xrightarrow{F_{\mathbb{T}}\eta_A} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} A \\
&= A \xrightarrow{F_{\mathbb{T}}\eta_A} TTA \xrightarrow{\mu_A} A \\
&= A \xrightarrow{\eta_A} TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{\mu_A} A \\
&= A \xrightarrow{\eta_A} TA \\
&= F_{\mathbb{T}}A \overset{1_{F_{\mathbb{T}}A}}{\dashrightarrow} F_{\mathbb{T}}A
\end{aligned}$$

Finally, note that

$$G_{\mathbb{T}}\varepsilon_{F_{\mathbb{T}}A} = TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA = \mu_A$$

and hence the adjunction induces the monad $\mathbb{T}$. □

Definition 5.8. Let $\mathcal{C}$ be a category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. We define a category, denoted $\mathbf{Adj}(\mathbb{T})$, whose objects are adjunctions $\left(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}\right)$ inducing $\mathbb{T}$ and morphisms $\left(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}\right) \to \left(\mathcal{C}' \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{D}'\right)$ are functors $K : \mathcal{D} \to \mathcal{D}'$ such that $KF = F'$ and $G'K = G$.

Theorem 5.9. *Let $\mathcal{C}$ be a category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. Then the Kliesli adjunction $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ is initial and the $\mathbb{T}$-algebra adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ is terminal in $\mathbf{Adj}(\mathbb{T})$.*

Proof. We shall first show that the $\mathbb{T}$-algebra adjunction is terminal in $\mathbf{Adj}(\mathbb{T})$. To this end, fix an adjunction $\left(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}\right)$ in $\mathbf{Adj}(\mathbb{T})$. We want to exhibit the existence of a unique morphism $(F \dashv G) \xrightarrow{K} (F^{\mathbb{T}} \dashv G^{\mathbb{T}})$. We first exhibit a functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ such that $KF = F^{\mathbb{T}}$ and $G^{\mathbb{T}}K = G$. Define the **algebra comparison functor** $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ by $KB = (GB, G\varepsilon_B)$ where ε is the counit of the adjunction $F \dashv G$. On morphisms we set $K(B \xrightarrow{g} B') = Gg$. We must verify that this is well-defined and functorial. $KB = (GB, G\varepsilon_B)$ is indeed a $\mathbb{T}$-algebra as $G\varepsilon_B$ is a morphism $TGB \to GB$, diagram 4 is satisfied from the triangular identities and diagram 5 and 6 follow from the naturality of ε. Furthermore, the functoriality of K follows from functoriality of G. Since $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is the forgetful functor, it is clear that $G^{\mathbb{T}}K = G$. Moreover, $KFA = (GFA, G\varepsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$ and $KF(A \xrightarrow{f} B) = GFf = Tf = F^{\mathbb{T}}f$ as desired.

To show uniqueness, suppose that $K' : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is another functor satisfying $G^{\mathbb{T}}K' = G$ and $K'F = F^{\mathbb{T}}$. From this we see that $K'B$ is of the form (GB, β_B) for some algebra structure $\beta_B : GFGB \to GB$ and that $\beta_{FA} = \mu_A$. Since $F \dashv G$ induces $\mathbb{T}$ we also have $\beta_{FA} = \mu_A = G\varepsilon_{FA}$ for all A. Now consider the diagram

$$\begin{array}{ccc} GFGFGB & \xrightarrow{GFG\varepsilon_B} & GFGB \\ \downarrow{\scriptstyle \mu_{GB}} & & \downarrow{\scriptstyle \beta_B} \\ GFGB & \xrightarrow{G\varepsilon_B} & GB \end{array}$$

which commutes since $G\varepsilon_B$ is a $\mathbb{T}$-algebra homomorphism. Similarly, the diagram with β_B replaced by $G\varepsilon_B$ is also commutative. We thus have that $(G\varepsilon_B) \circ (GFG\varepsilon_B) = (\beta_B) \circ (GFG\varepsilon_B)$. But $GFG\varepsilon_B$ is split epic by the triangular identities and so $G\varepsilon_B = \beta_B$ whence $K = K'$ and $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ is terminal.

We now show that the Kleisli adjunction is initial in $\mathbf{Adj}(\mathbb{T})$. That is, we want to exhibit the existence of a unique morphism $(F_{\mathbb{T}} \dashv G_{\mathbb{T}}) \xrightarrow{K} (F \dashv G)$. Define the **Kleisli comparison functor** by $KA = FA$ and $K(A \overset{f}{\dashrightarrow} B) = FA \xrightarrow{Ff} FGFB \xrightarrow{\varepsilon_{FB}} FB$. We shall first verify that this is functorial. Suppose we are given a composition $A \overset{f}{\dashrightarrow} B \overset{g}{\dashrightarrow} C$. Then by the naturality of ε

$$\begin{aligned} K(A \overset{f}{\dashrightarrow} B \overset{g}{\dashrightarrow} C) &= FA \xrightarrow{Ff} FGFB \xrightarrow{FGFg} FGFGFC \xrightarrow{FG\varepsilon_{FC}} FGFC \xrightarrow{\varepsilon_{FC}} FC \\ &= FA \xrightarrow{Ff} FGFB \xrightarrow{FGFg} FGFGFC \xrightarrow{\varepsilon_{FGFC}} FGFC \xrightarrow{\varepsilon_{FC}} FC \\ &= FA \xrightarrow{Ff} FGFB \xrightarrow{\varepsilon_{FB}} FB \xrightarrow{Fg} FGFC \xrightarrow{\varepsilon_{FC}} FC \\ &= (Kg)(Kf) \end{aligned}$$

We next claim that K is a morphism $(F_{\mathbb{T}} \dashv G_{\mathbb{T}}) \xrightarrow{K} (F \dashv G)$. To this end, we must show that $KF_{\mathbb{T}} = F$ and $GK = G_{\mathbb{T}}$. For the latter equality, observe that $GKA = GFA = TA = G_{\mathbb{T}}A$ and $GK(A \overset{f}{\dashrightarrow} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB = G_{\mathbb{T}}f$ as desired. For the former equality, we have that $KF_{\mathbb{T}}A = FA$ and

$$\begin{aligned} KF_{\mathbb{T}}(A \xrightarrow{f} B) &= FA \xrightarrow{Ff} FB \xrightarrow{F\eta_B} FGFB \xrightarrow{\varepsilon_{FB}} FB \\ &= FA \xrightarrow{Ff} FB \end{aligned}$$

as desired and so K is a morphism in $\mathbf{Adj}(\mathbb{T})$. It remains to show that K is unique. Suppose that $K' : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ is any other morphism satisfying $K'F_{\mathbb{T}} = F$ and $GK' = G_{\mathbb{T}}$. Clearly,

$K'A = FA = KA$ for all A. Furthermore,

$$\begin{aligned} GK'(TA \overset{1_{TA}}{\dashrightarrow} A) &= G_{\mathbb{T}}(TA \overset{1_{TA}}{\dashrightarrow} A) \\ &= TTA \xrightarrow{T1_{TA}} TTA \xrightarrow{\mu_A} TA \\ &= TTA \xrightarrow{\mu_A} TA \\ &= GFGFA \xrightarrow{G\varepsilon_{FA}} GFA \end{aligned}$$

and so $K'(TA \overset{1_{TA}}{\dashrightarrow} A) = \varepsilon_{FA}$. Now note that given any morphism $A \overset{f}{\dashrightarrow} B$ we can write it in the form $A \overset{F_{\mathbb{T}}f}{\dashrightarrow} TB \overset{1_{TB}}{\dashrightarrow} B$ and so

$$\begin{aligned} K'(A \overset{f}{\dashrightarrow} B) &= K'(A \overset{F_{\mathbb{T}}f}{\dashrightarrow} TB \overset{1_{TB}}{\dashrightarrow} B) \\ &= K'(TB \overset{1_{TB}}{\dashrightarrow} B)K'(A \overset{F_{\mathbb{T}}f}{\dashrightarrow} TB) \\ &= (\varepsilon_{FB}) \circ (K'F_{\mathbb{T}f}) \\ &= (\varepsilon_{FB}) \circ (Ff) \\ &= Kf \end{aligned}$$

and so $K = K'$ and the Kleisli adjunction is initial in $\mathbf{Adj}(\mathbb{T})$. □

Theorem 5.10. *Let $\mathcal{C}$ and J be categories and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. Then*

1. *The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates all limits which exist in $\mathcal{C}$.*
2. *If T preserves colimits of shape J then $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates them.*

Proof. Let $D : J \to \mathcal{C}^{\mathbb{T}}$ be a diagram and write $D(j) = (GD(j), \delta_j)$ for all $j \in J$. Let $(L, (\lambda_j : L \to GD(j)))$ be a limit cone for GD. Observe that the composites $TL \xrightarrow{T\lambda_j} TGD(j) \xrightarrow{\delta_j} GD(j)$ form a cone over GD. Indeed for all $\alpha : j \to j'$, the diagram

$$\begin{array}{ccccc} & & TL & & \\ & \swarrow^{T\lambda_j} & & \searrow^{T\lambda_{j'}} & \\ TGD(j) & & \xrightarrow{TGD(\alpha)} & & TGD(j') \\ \downarrow{\scriptstyle \delta_j} & & & & \downarrow{\scriptstyle \delta_{j'}} \\ GD(j) & & \xrightarrow{GD(\alpha)} & & GD(j') \end{array}$$

commutes since the $D(\alpha)$ are $\mathbb{T}$-algebra homomorphisms and L is a cone over GD. This implies that there exists a unique $\lambda : TL \to L$ such that the diagram

$$\begin{array}{ccc} TL & \xrightarrow{\lambda} & L \\ \downarrow{\scriptstyle T\lambda_j} & & \downarrow{\scriptstyle \lambda_j} \\ TGD(j) & \xrightarrow{\delta_j} & GD(j) \end{array}$$

commutes. We now claim that λ is an algebra structure. To this end, we need to first show that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\eta_L} & TL \\ & \searrow^{1_L} & \downarrow{\scriptstyle \lambda} \\ & & L \end{array}$$

commutes. By the naturality of η and the fact that $D(j)$ is a $\mathbb{T}$-algebra, we have that for all j

$$\lambda_j \circ \lambda \circ \eta_L = \delta_j \circ T\lambda_j \circ \eta_L = \delta_j \circ \eta_{GD(j)} \circ \lambda_j = 1_{GD(j)} \circ \lambda_j = \lambda_j$$

We see that both $\lambda \circ \eta_L$ and 1_L are factorisations of the limit cone L through itself and so they must be equal. We must next show that the diagram

$$\begin{array}{ccc} TTL & \xrightarrow{T\lambda} & TL \\ \downarrow{\scriptstyle \mu_L} & & \downarrow{\scriptstyle \lambda} \\ TL & \xrightarrow{\lambda} & L \end{array}$$

commutes. On one hand, we have that

$$\begin{aligned} \lambda_j \circ \lambda \circ T\lambda &= \delta_j \circ T\lambda_j \circ T\lambda \\ &= \delta_j \circ T(\lambda_j \circ \lambda) \\ &= \delta_j \circ T(\delta_j \circ T\lambda_j) \\ &= \delta_j \circ T\delta_j \circ TT\lambda_j \end{aligned}$$

On the other hand, the naturality of μ implies that

$$\begin{aligned} \lambda_j \circ \lambda \circ \mu_L &= \delta_j \circ T\lambda_j \circ \mu_L \\ &= \delta_j \circ \mu_{GD(j)} \circ TT\lambda_j \end{aligned}$$

Since the δ_j are algebra structures, we have that $\delta_j \circ T\delta_j = \delta_j \circ \mu_{GD(j)}$ and so $\lambda_j \circ \lambda \circ T\lambda = \lambda_j \circ \lambda \circ \mu_L$. These are both factorisations of the same cone through the limit so we must have that they are equal.

Finally, we claim that $((L, \lambda), (\lambda_j))$ is a limit for the diagram D. It is clearly a cone as it's image under G is a cone over $GD(j)$ and the λ_j are $\mathbb{T}$-algebra homomorphisms. To show that it is terminal, suppose we are given another cone $((M, \nu), (\nu_j))$ over $D(j)$. Then its image under G admits a unique factorisation $\nu_j = \lambda_j \varphi$ for some morphism $\varphi : M \to L$. It suffices to show that φ is a homomorphism of $\mathbb{T}$-algebras. In other words, we must show that the diagram

$$\begin{array}{ccc} TM & \xrightarrow{T\varphi} & TL \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \lambda} \\ M & \xrightarrow{\varphi} & L \end{array}$$

On one hand we have

$$\begin{aligned} \lambda_j \circ \lambda \circ T\varphi &= \delta_j \circ T\lambda_j \circ T\varphi \\ &= \delta_j \circ T(\lambda_j \circ \varphi) \\ &= \delta_j \circ T(\nu_j) \end{aligned}$$

On the other hand, we have

$$\lambda_j \circ \varphi \circ \nu = \nu_j \circ \nu$$

But ν_j is a $\mathbb{T}$-algebra homomorphism and so these must be equal. They are thus two factorisations of the same cone through the limit whence $\lambda \circ T\varphi = T(\nu_j)$ as desired. $\square$

Definition 5.11. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Given an adjunction $F \dashv G$ between them, we say that $(F \dashv G)$ is **monadic** if the algebra comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is part of an equivalence of categories where $\mathbb{T}$ is the monad induced by $(F \dashv G)$. Moreover, we say that a functor $G : \mathcal{D} \to \mathcal{C}$ is **monadic** if it has a left adjoint F and $F \dashv G$ is monadic.

Lemma 5.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F \dashv G$ an adjunction between them with counit ε inducing a monad $\mathbb{T}$. Suppose that for all $\mathbb{T}$-algebras (A, α), the pair*

$$FGFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$$

has a coequalizer in $\mathcal{D}$. Then the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ has a left adjoint L.

Proof. Given a $\mathbb{T}$-algebra (A, α), define $L(A, \alpha)$ to be the coequaliser of the parallel pair

$$FGFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$$

Then, given any homomorphism of $\mathbb{T}$-algebras $f : (A, \alpha) \to (B, \beta)$ we have the following diagram

$$\begin{array}{ccccc} FGFA & \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} & FA & \longrightarrow & L(A,\alpha) \\ \downarrow{\scriptstyle FGFf} & & \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Lf} \\ FGFB & \underset{\varepsilon_{FB}}{\overset{F\beta}{\rightrightarrows}} & FB & \longrightarrow & L(B,\beta) \end{array}$$

By the universal property of coequalisers, there exists a unique morphism $Lf : L(A, \alpha) \to L(B, \beta)$ extending the diagram to a commutative diagram. The uniqueness of this morphism ensures that L is functorial.

To show that $L \dashv K$, it suffices to show that $\mathcal{D}(L(A, \alpha), B) \cong \mathcal{C}^{\mathbb{T}}((A, \alpha), KB)$. Observe that morphisms $L(A, \alpha) \to B$ are in one-to-one correspondence with morphisms $f : FA \to B$ that coequalise $F\alpha$ and ε_{FA}. In other words, $f \circ F\alpha = f \circ \varepsilon_{FA}$. But such morphisms correspond to morphisms $f' : A \to GB$ such that $f'\alpha = Gf$. Since $f = \varepsilon_B \circ Ff'$ we then have that $f'\alpha = G\varepsilon_B \circ GFf'$. But this is exactly what it means for f' to be a $\mathbb{T}$-algebra homomorphism $(A, \alpha) \to (GB, G\varepsilon_B) = KB$. We omit the proof that these natural transformations are natural in (A, α) and B. $\square$

Definition 5.13. Let $\mathcal{C}$ be a category.

1. We say that a parallel pair $f, g : A \rightrightarrows B$ of morphisms in $\mathcal{C}$ is **reflexive** if there exists a morphism $r : B \to A$ such that $fr = gr = 1_B$.

2. By a **split coequaliser** diagram, we mean a diagram of the form

$$\begin{array}{ccccc} A & \underset{g}{\overset{f}{\rightrightarrows}} & B & \overset{h}{\longrightarrow} & C \\ & \overset{t}{\longleftarrow} & & \overset{s}{\longleftarrow} & \end{array}$$

   such that $hf = hg, hs = 1_C, gt = 1_B$ and $ft = sh$.

3. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. We say that a parallel pair $f, g : A \rightrightarrows B$ in $\mathcal{D}$ is **G-split** if there exists a split coequaliser

$$GA \underset{Gg}{\overset{Gf}{\rightrightarrows}} GB \xrightarrow{h} C \qquad \underset{t}{GB \curvearrowright GA} \qquad \underset{s}{C \curvearrowright GB}$$

in $\mathcal{C}$.

Lemma 5.14. *Let $\mathcal{C}$ be a category. Then any split coequaliser in $\mathcal{C}$ is a coequaliser.*

Proof. Suppose we are given a split coequaliser diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C \qquad \underset{t}{B \curvearrowright A} \qquad \underset{s}{C \curvearrowright B}$$

Let $k : B \to D$ be any other morphism satisfying $kf = kg$. We claim that k must factor through h. Indeed, $k = kgt = kft = ksh$ and is the unique such factorisation as h is split epic. □

Theorem 5.15 (Precise Monadicity Theorem)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $G : \mathcal{D} \to \mathcal{C}$ is monadic if and only if G has a left adjoint and creates coequalisers of G-split pairs.*

Proof. Assume that G is monadic and let $\mathbb{T} = (T, \eta, \mu)$ be the monad induced by the adjunction. Then by definition it has a left adjoint so we just need to show that G creates coequalisers of G-split pairs. Since $\mathcal{D} \cong \mathcal{C}^{\mathbb{T}}$, it suffices to show that the forgetful functor $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates coequalisers of $G^{\mathbb{T}}$-split pairs. To this end, suppose that $f, g : (A, \alpha) \to (B, \beta)$ is a G-split pair and let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C \qquad \underset{t}{B \curvearrowright A} \qquad \underset{s}{C \curvearrowright B}$$

be a split coequaliser diagram for it in $\mathcal{C}$. Any functor will clearly preserve a split coequaliser and so, in particular, T preserves split coequalisers. Theorem 5.10 then implies that $G^{\mathbb{T}}$ creates coequalisers of G-split pairs.

Conversely, suppose that G has a left adjoint and creates coequalisers of G-split pairs. Let $\mathbb{T} = (T, \eta, \mu)$ be the monad induced by the adjunction $F \dashv G$. We want to show that $\mathcal{D} \cong \mathcal{C}^{\mathbb{T}}$. We do this by constructing a weak inverse $L : \mathcal{C}^{\mathbb{T}} \to \mathcal{D}$ to the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$. Fix a $\mathbb{T}$-algebra (A, α). Observe that the parallel pair

$$FGFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$$

is G-split since

$$GFGFA \underset{G\varepsilon_{FA}}{\overset{GF\alpha}{\rightrightarrows}} GFA \xrightarrow{\alpha} A \qquad \underset{\eta_{GFA}}{GFA \curvearrowright GFGFA} \qquad \underset{\eta_A}{A \curvearrowright GFA}$$

is a split coequaliser. Indeed, $\alpha \circ (GF\alpha) = \alpha \circ (G\varepsilon_{FA})$ by diagram 5, $\alpha \circ \eta_A = 1_A$ by diagram 4, $(G\varepsilon_{FA}) \circ \eta_{GFA} = 1_{GFA}$ by the triangular identities and $(GF\alpha) \circ (\eta_{GFA}) = \eta_A \circ \alpha$ by the naturality of η. By hypothesis, G creates coequalisers of G-split pairs so Lemma 5.12 implies that K has a left adjoint $L : \mathcal{C}^{\mathbb{T}} \to \mathcal{D}$. We claim that $KL \xrightarrow{\sim} 1_{\mathcal{C}^{\mathbb{T}}}$ and $LK \xrightarrow{\sim} 1_{\mathcal{D}}$.

For the former, fix a $\mathbb{T}$-algebra (A, α). Then $KL(A, \alpha) = (GL(A, \alpha), G\varepsilon_{L(A,\alpha)})$ where $L(A, \alpha)$ is the coequaliser in the diagram

$$FGFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA \xrightarrow{\theta} L(A, \alpha)$$

Observe that

$$GFGFA \underset{\varepsilon_{GFA}}{\overset{GF\alpha}{\rightrightarrows}} GFA \xrightarrow{G\theta} GL(A,\alpha)$$

is also a coequaliser diagram since G creates and, in particular, preserves limits. By uniqueness of limits, we must have that $G\theta = \alpha$ and $GL(A,\alpha) \cong A$. We must now show that $G\varepsilon_{L(A,\alpha)} = \alpha$. Note that it suffices to show that $\theta = \varepsilon_{L(A,\alpha)}$. By definition, θ coequalises the maps $F\alpha$ and ε_{FA} and so $\theta \circ F\alpha = \theta \circ \varepsilon_{FA}$. By the naturality of ε, it follows that $\theta \circ F\alpha = \varepsilon_{L(A,\alpha)} \circ FG\theta = \varepsilon_{L(A,\alpha)} \circ F\alpha$. Composing both sides on the right by $F(\eta_A)$ and using diagram 5 yields $\theta = \varepsilon_{L(A,\alpha)}$ as desired.

To show that $LK \overset{\sim}{\to} 1_{\mathcal{D}}$, let $B \in \operatorname{ob}\mathcal{D}$. Then $LKB = L(GB, G\varepsilon_B)$. We have two coequaliser diagrams

$$FGFGB \underset{\varepsilon_{FGB}}{\overset{FG\varepsilon_B}{\rightrightarrows}} FGB \longrightarrow L(GB, G\varepsilon_B) \qquad FGFGB \underset{\varepsilon_{FGB}}{\overset{FG\varepsilon_B}{\rightrightarrows}} FGB \xrightarrow{\varepsilon_B} B$$

as $FG\varepsilon_B$ and ε_{FGB} is a G-split pair. We must therefore have that $L(GB, G\varepsilon_B) \cong B$. □

Theorem 5.16 (Crude Monadicity Theorem)**.** *Let $\mathcal{C}, \mathcal{D}$ be categories and $G : \mathcal{D} \to \mathcal{C}$ a functor. Then G is monadic if G has a left adjoint, preserves reflexive coequalisers and reflects isomorphisms.*

Proof. Follows the same proof as the backwards direction of the Precise Monadicity Theorem. □

Lemma 5.17. *Suppose that*

$$A_1 \underset{g_1}{\overset{f_1}{\rightrightarrows}} B_1 \xrightarrow{h_1} C_1, \quad r_1 : B_1 \to A_1 \qquad\qquad A_2 \underset{g_2}{\overset{f_2}{\rightrightarrows}} B_2 \xrightarrow{h_2} C_2, \quad r_2 : B_2 \to A_2$$

are reflexive coequaliser diagrams in **Set**. *Then*

$$A_1 \times A_2 \underset{g_1 \times g_2}{\overset{f_1 \times f_2}{\rightrightarrows}} B_1 \times B_2 \xrightarrow{h_1 \times h_2} C_1 \times C_2$$

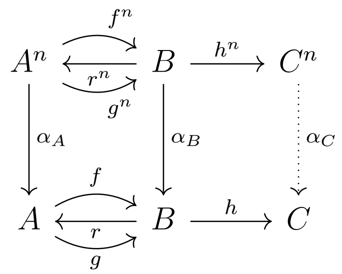
is a coequaliser diagram.

Proof. Without loss of generality, we may assume that $C_i = B_i/\sim$ where $b_i \sim b_i'$ if and only if there exists a chain of elements $b_i = x_1, \ldots, x_n = b_i'$ such that each $\{ x_j, x_{j+1} \}$ is of the form $\{ f_i(y_j), g_i(y_j) \}$ for some $y_j \in A_i$. If we have chains linking b_1 to b_1' and b_2 to b_2' then we can link (b_1, b_2) by first linking it to (b_1', b_2). □

Example 5.18. The forgetful functors $\mathbf{Grp} \to \mathbf{Set}, \mathbf{Rng} \to \mathbf{Set}$ and $\mathbf{Mod}_R \to \mathbf{Set}$ are all monadic. Indeed, suppose we are given a reflexive coequaliser

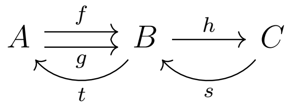
$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C, \quad r : B \to A$$

where A and B have some finitary algebraic structure provided by an n-ary operation α. Consider the diagram



So C inherits a finitary algebraic structure making h into a homomorphism and C a coequaliser in the corresponding algebraic category.

Example 5.19. Any reflection is monadic. Indeed, suppose that we are given a reflective full subcategory $G : \mathcal{D} \to \mathcal{C}$. By definition, G has a left adjoint so we just need to show that G creates split coequalisers. To this end, suppose that we have a parallel pair $f, g : A \rightrightarrows B$ in $\mathcal{D}$ that has a split coequaliser diagram

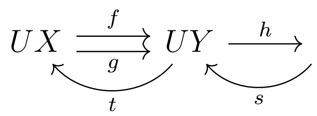


Clearly $t \in \operatorname{mor} \mathcal{D}$ whence $ft = sh \in \operatorname{mor} \mathcal{D}$. Observe that $shs = s1_C = s = 1_B s$ and so s is an equaliser for sh and 1_B in $\mathcal{C}$. Since the inclusion functor creates all limits in $\mathcal{C}$, we must have that, up to isomorphism, (C, h) is a coequaliser in $\mathcal{D}$.

Example 5.20. Consider the category **tfAbGrp** of torsion free abelian groups. The forgetful functor **tfAbGrp** $\to$ **AbGrp** is monadic and so is the inclusion **tfAbGrp** $\to$ **AbGrp** as it is a reflection. However, the composite **tfAbGrp** $\to$ **AbGrp** $\to$ **Set** is not monadic since the monad it induces on **Set** is isomorphic to that induced by **AbGrp** $\to$ **Set** so its category of algebras is equivalent to **AbGrp**.

Example 5.21. The forgetful functor **Top** $\to$ **Set** isn't monadic. It has a left adjoint but the induced monad is $(1_{\mathbf{Set}}, 1_{1_{\mathbf{Set}}}, 1_{1_{\mathbf{Set}}})$ and so its category of algebras is isomorphic to **Set**.

Example 5.22. The forgetful functor U : **KHaus** $\to$ **Set** is monadic. It has a left adjoint **Set** $\xrightarrow{D}$ **Top** $\xrightarrow{\beta}$ **KHaus** where β is the Stone-Čech compactification. It thus suffices to show that U creates coequalisers of U-split pairs. Let $f, g : X \rightrightarrows Y$ be a parallel pair in **KHaus** such that



Define an equivalence relation on UY with $y_1 \sim y_2$ if $h(y_1) = h(y_2)$. We claim that $UY/ \ = Z$ equipped with the quotient topology is compact Hausdorff with h continuous. h is the canonical surjective mapping so it is clearly continuous. Furthermore, the quotient of a compact space is always compact. Hence it remains to show that Z is Hausdorff. By a result from point-set topology, if $R \subseteq Y \times Y$ is an equivalence relation then Y/R is Hausdorff if and only if R is closed in $Y \times Y$. Hence it suffices to show that

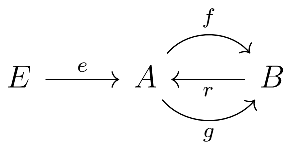
$$R = \{ (y, y') \in Y \mid h(y) = h(y') \}$$

is closed in $Y \times Y$. But h coequalises f and g so (y, y') is of the form $(g(x), g(x'))$ such that $f(x) = f(x')$ for some $(x, x') \in X \times X$. Hence R is the image under $X \times X \xrightarrow{g \times g} Y \times Y$ of

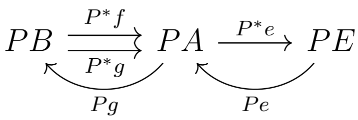
$$S = \{ (x, x') \mid f(x) = f(x') \}$$

Observe that S is closed in X since f is continuous and Y is Hausdorff and so S is compact whence R is compact. This implies that R is closed as desired.

Example 5.23. The contravariant power-set functor $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is monadic. Indeed, it self is adjoint and it reflects isomorphisms so it suffices to show that it sends coreflexive equalisers in **Set** to reflexive coequalisers. Let



be a coreflexive equaliser diagram. In other words, $rf = rg = 1_A$. Observe that if $f(a) = g(a')$ then $a = a' = r(f(a))$ and so the images of f and g coincide on the image of E. The same argumentation also shows that f and g are injective. Now consider the diagram

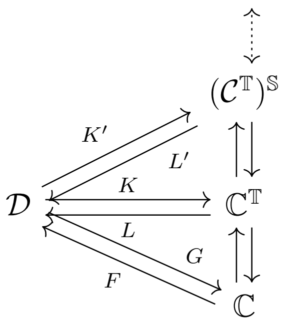


We claim that this is a split coequaliser diagram. Since P^* is a functor, we have that $(P^*e)(P^*f) = (P^*e)(P^*g)$. Moreover, since e and g are monic, we have $(P^*e)(Pe) = 1_{PE}$ and $(P^*g)(Pg) = 1_{PA}$. Finally,

$$(P^*f)(Pg)(A') = A' \cap \mathrm{im}(e) = (Pe)(P^*e)(A')$$

for all $A' \subseteq A$.

Definition 5.24. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F \dashv G$ an adjunction between them. Suppose that $\mathcal{D}$ has reflexive coequalisers. By the **monadic tower** of $F \dashv G$, we mean the diagram



where $\mathbb{T}$ is the monad induced by $F \dashv G$, $\mathbb{S}$ is the monad induced by $K \dashv L$ and $L^{(n)} \dashv K^{(n)}$ is the adjunction comprised of the algebra comparison functor and its left adjoint. We say that $F \dashv G$ has **monadic length** n if we reach an equivalence after n steps.

Example 5.25. If $G : \mathcal{D} \to \mathcal{C}$ is a reflective full subcategory with left adjoint F then $F \dashv G$ has monadic length 2.

Example 5.26. Let D be the left adjoint to the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$. Then $D \dashv U$ has monadic length ∞.

6 Regular Categories

Definition 6.1. Let $\mathcal{C}$ be a category and $(A \xrightarrow{f} B) \in \operatorname{mor} \mathcal{D}$ a morphism. By the **image** of f, we mean the smallest subobject of B through which f factors if it exists. We say that $\mathcal{C}$ **has images** if every $f \in \operatorname{mor} \mathcal{C}$ has an image. Furthermore, we say that f is a **cover** if its image is 1_B. We shall write $A \overset{f}{\twoheadrightarrow} B$ to indicate that f is a cover.

Definition 6.2 ($\star$)**.** Let $\mathcal{C}$ be a category and $A \xrightarrow{f} B$ an epimorphism. We say that f is a **strong epimorphism** if given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & \overset{t}{\nearrow} & \downarrow{\scriptstyle k} \\ B & \xrightarrow{h} & D \end{array}$$

with k monic, there exists $t : B \to C$ such that $g = tf$ and $h = kt$.

Lemma 6.3. *Let $\mathcal{C}$ be a category with all finite limits. Then the covers in $\mathcal{C}$ coincide with strong epimorphisms.*

Proof. Suppose that f is strong epic. We claim that f is a cover. To this end, suppose that f factors through a subobject of B, say $f = gh$ where $g : C \rightarrowtail B$ is monic. It suffices to show that, in fact, g is an isomorphism. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{1_B} & B \end{array}$$

Since f is strong epic, there exists a $t : B \to C$ such that $1_B = gt$. This means that g is split epic. But it is also monic whence g is an isomorphism as desired.

Conversely, suppose that $f : A \twoheadrightarrow B$ is a cover. We first claim that f is epic. To this end, suppose that $gf = hf$ for a parallel pair $g, h : B \rightrightarrows C$. Since $\mathbb{C}$ has all finite limits, it has equalisers and so f factors through the equaliser $e : E \to B$ of g and h, say $f = ze$ for some $z : E \to B$. But every equaliser is monic and f is a cover so we must have that e is an isomorphism whence $g = h$ and f is epic.

Now suppose that we are given the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow{h} & D \end{array}$$

with k monic. We need to exhibit a $t : B \to C$ that fills in the diagonal of this diagram. We may form the pullback

$$\begin{array}{ccc} P & \xrightarrow{n} & C \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow{h} & D \end{array}$$

Since monics are stable under pullback, we have that m is monic. But then f factors through the subobject $m : P \to B$ so we must have that m is an isomorphism. $B \xrightarrow{nm^{-1}} C$ is thus the desired diagonal fill in for the diagram. $\square$

Definition 6.4. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is **regular** if $\mathcal{C}$ has all finite limits and images and in which all strong epimorphisms are stable under pullback. Furthermore, we say that a functor $G : \mathcal{C} \to \mathcal{D}$ between regular categories is **regular** if it preserves all finite limits and strong epimorphisms.

Example 6.5. Set is regular.

Example 6.6. If $\mathcal{D}$ is regular then so is $[\mathcal{C}, \mathcal{D}]$ for any $\mathcal{C}$ with finite limits and images constructed pointwise.

Example 6.7. If $\mathcal{C}$ is regular and $\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathcal{C}$ such that T preserves strong epimorphisms then $\mathbb{C}^{\mathbb{T}}$is regular with finite limits and images created by the forgetful functor $\mathbb{C}^{\mathbb{T}} \to \mathbb{C}$. In particular, any category monadic over **Set** is regular.

Example 6.8. Top has images but isn't regular. However, $\mathbf{Top}^{\mathrm{op}}$ is regular.

Definition 6.9. Let $\mathcal{C}$ be a category that has pullbacks and $f : A \to B$ a morphism in $\mathcal{C}$. We define the **kernel pair** of f to be the pullback of f against itself. In other words, it is the limit of the diagram

$$\begin{array}{ccc} & & A \\ & & \big\downarrow f \\ A & \xrightarrow{\;f\;} & B \end{array}$$

Proposition 6.10. *Let $\mathcal{C}$ be a regular category. Then the strong epimorphisms in $\mathcal{C}$ coincide with the regular epimorphisms.*

Proof. Suppose that $f : B \to C$ is a regular epimorphism. That is to say, f occurs as the coequaliser of some parallel pair $g, h : A \rightrightarrows B$. We first claim that f is epic. To this end, suppose that we are given morphisms $x, y : B \to D$ such that $xf = yf$. We want to prove that $x = y$. Note that

$$xf = yf \implies xfg = yfg \implies xfg = yfh$$

and so xf and yf coequalise g and h. They must therefore factorise uniquely through f. Clearly the unique factorisations are given by x and y respectively. But these are two factorisations of the same cocone through the coequaliser so we must have that $x = y$ whence f is epic. We must now show that f is a strong epimorphism. Hence consider the diagram

$$\begin{array}{ccccc} A & \overset{g}{\underset{h}{\rightrightarrows}} & B & \xrightarrow{\;f\;} & C \\ & & \big\downarrow x & & \big\downarrow y \\ & & D & \xrightarrow{\;k\;} & E \end{array}$$

with k monic. We need to show that there exists a $t : C \to D$ such that $x = tf$ and $y = kt$. Since f coequalises g and h, we have that $fg = fh$. Post composing by y gives $yfg = yfh$. By commutativity of the diagram, we have that $kxg = kxh$ whence $xg = xh$ since k is monic. x thus coequalises g and h and there exists a unique $t : C \to D$ such that $x = tf$. To see that $y = kt$, note that $yx = ktf$ and so $yf = ktf$. But f is epic and so $y = kt$ as desired.

Now suppose that $f : B \to C$ is a strong epimorphism. We need to show that f occurs as the coequaliser of some two morphisms. Let $a, b : R \to A$ be the kernel pair of R whose existence is guaranteed by the fact that $\mathcal{C}$ is regular and so, in particular, has all finite limits. We claim that f is a coequaliser for a and b. By commutativity of the kernel pair, we have that $fa = fb$ so we just need to show that, given any $g : B \to C$ that coequalises a and b we have that g factors uniquely through f. We may form the image I of (f, g) so that we have $A \xrightarrow{h} I \overset{(k,l)}{\rightarrowtail} B \times C$. It suffices to show that k is an isomorphism. Indeed, if that were the case, we would have that $lk^{-1}f = lk^{-1}kh = lh = g$ with uniqueness following from the fact that f is epic.

Observe that k is strong epic since $kh = f$ is strong epic. It thus suffices to show that k is monic. Suppose we are given morphisms $x, y : D \rightrightarrows I$ satisfying $kx = ky$. We may form the pullback

$$\begin{array}{ccc} E & \xrightarrow{\quad m \quad} & D \\ \downarrow{\scriptstyle (n,p)} & & \downarrow{\scriptstyle (x,y)} \\ A \times A \xrightarrow{(1_A,h)} A \times I & \xrightarrow{(h,1_A)} & I \times I \end{array}$$

Since strong epimorphisms are stable under pullback in $\mathcal{C}$ and $(1_A, h)$ and $(h, 1_A)$ are strong epimorphisms, it follows that m is a strong epimorphism. Note that

$$fn = khn = kxm = kym = khp = fp$$

and so (n, p) factors through (a, b), say by $E \xrightarrow{q} R$. We have that $kha = khb$ and (k, l) is monic so $ha = hb$. Then $xm = hn = haq = hbq = hp = ym$. But m is epic and so $x = y$ whence k is monic. □

Definition 6.11. Let $\mathcal{C}$ be a category that has all finite limits and $a, b : R \rightrightarrows A$ a parallel pair in $\mathcal{C}$.

1. We say that (a, b) is a **relation** if $R \xrightarrow{(a,b)} A \times A$ is monic.
2. We say that (a, b) is **reflexive** if there exists $A \xrightarrow{r} R$ such that $ar = br = 1_A$.
3. We say that (a, b) is **symmetric** if there exists $A \xrightarrow{s} R$ such that $as = b$ and $bs = a$.
4. We say that (a, b) is **transitive** if, given the pullback

$$\begin{array}{ccc} T & \xrightarrow{q} & R \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle a} \\ R & \xrightarrow{b} & A \end{array}$$

there exists $t : T \to R$ such that $at = ap$ and $bt = bq$.

Moreover, if all the above hold then we say that (a, b) is an **equivalence relation**.

Proposition 6.12. *Let $\mathcal{C}$ be a category with all finite limits and $f : A \to B$ be a morphism in $\mathcal{C}$. Then the kernel pair of f is an equivalence relation.*

Proof. Let (a, b) be the kernel pair of f. We first show that (a, b) is a relation. Suppose that we are given a parallel pair $x, y : D \rightrightarrows R$ such that $(a, b)x = (a, b)y$. We need to show that $x = y$. By the universal property of the pullback, there exists a unique $u : D \to R$ such that $au = ax$ and $bu = by$. But then u and x are two factorisations of the same cone (which is guaranteed to be a cone by the fact that $(a, b)x = (a, b)y$) through the pullback so we must have that $u = x$. Similarly, $u = y$ and so $x = y$.

We next show that (a, b) is reflexive. Consider the cone over the parallel pair whose apex is A and whose legs are 1_A. Then this cone completes the parallel pair to a commutative diagram and so it must factor through the pullback. In particular, there exists a (unique) $r : A \to R$ such that $ar = br = 1_A$.

We now show that (a, b) is symmetric. This is easy to see by considering the cone that is the mirror image of the pullback in the diagonal.

Finally, to show transitivity, suppose we are given such a pullback square. Then consider the diagram

$$\begin{array}{ccccccc}
 & & & & A & \xrightarrow{f} & B \\
 & & & & \uparrow{\scriptstyle a} & & \uparrow{\scriptstyle f} \\
 & & T & \xrightarrow{q} & R & \xrightarrow{b} & A \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle a} & & \\
A & \xleftarrow[a]{} & R & \xrightarrow{b} & A & & \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} & & & & \\
B & \xleftarrow[f]{} & A & & & &
\end{array}$$

From this we see that there are morphisms $ap : T \to A$ and $bq : T \to A$. We first claim that these form a cone over the parallel pair $f, f : A \rightrightarrows B$. In other words, we have to show that $fap = fbq$. We have that

$$fap = fbp = faq = fbq$$

by the commutativity of the diagram. This cone must factor through the pullback so there exists some (unique) $t : T \to R$ such that $ap = at$ and $bq = bt$ as desired. □

Definition 6.13. Let $\mathcal{C}$ be a category that has all finite limits and (a, b) an equivalence relation in $\mathcal{C}$. We say that (a, b) is **effective** if it occurs as the kernel pair of some morphism in $\mathcal{C}$. Furthermore, if $\mathcal{C}$ is regular then we say that $\mathcal{C}$ is **effective regular** (or **Barr exact**) if every equivalence relation in $\mathcal{C}$ is effective.

Example 6.14. Set is effective regular.

Example 6.15. $\mathcal{D}$ is effective regular if and only if $[\mathcal{C}, \mathcal{D}]$ is effective regular.

Example 6.16. Let $\mathcal{C}$ be effective regular and $\mathbb{T}$ a monad on $\mathcal{C}$ such that T preserves strong epimorphisms. Then $\mathcal{C}^{\mathbb{T}}$ is effective regular.

Example 6.17. tfAbGrp is regular but not effective. Indeed, the equivalence relation $\{ (a, b) \in \mathbb{Z}^2 \mid a \equiv b \pmod 2 \}$ is a non-effective equivalence relation on $\mathbb{Z}$.

Definition 6.18. Let $\mathcal{C}$ be a regular category and $A \in \operatorname{ob} \mathcal{C}$ an object.

1. We define the **support** of A, denoted σA to be the image of the unique morphism $A \to 1$.

2. We say that A is **well-supported** if $\sigma A \cong 1$ (in other words, $A \to 1$ is strong epic).

3. We say that A is **totally-supported** if all its objects are well-supported.

4. We say that an object $0 \in \mathcal{C}$ is **strict** if any morphism $A \to 0$ is an isomorphism.

Moreover, we say that $\mathcal{C}$ is **almost totally supported** if every object of $\mathcal{C}$ is either well-supported or strict.

Lemma 6.19. *Let $\mathcal{C}$ be a regular category and $0 \in \mathcal{C}$ a strict object. Then 0 is initial.*

Proof. We need to show that, given an object $A \in \operatorname{ob}\mathcal{C}$, there exists a unique morphism $0 \to A$. We first show existence. Consider the product $0 \times A$. Then we have an isomorphism $0 \times A \xrightarrow{\pi_1} 0$ and so $0 \xrightarrow{\pi_1^{-1}} 0 \times A \xrightarrow{\pi_2} A$ exists. To show uniqueness, suppose we are given a parallel pair $f, g : 0 \rightrightarrows A$. Then the equaliser of f and g is a morphism $0 \xrightarrow{e} A$ such that $fe = ge$. But 0 is strict so f must b an isomorphism. In particular, it is epic and so $f = g$. □

Theorem 6.20 (Barr's Embedding Theorem)**.** *Let $\mathcal{C}$ be a small regular category. Then there exists a small category $\mathcal{D}$ and a fully faithful regular functor $\mathcal{C} \to [\mathcal{D}, \mathbf{Set}]$. Furthermore, if $\mathcal{C}$ is almost totally supported then $\mathcal{D}$ can be taken to be a monoid.*

We shall only prove one aspect of this theorem:

Theorem 6.21. *Let $\mathcal{C}$ be a small almost totally supported regular category. Then there exists an isomorphism-reflecting regular functor $F : \mathcal{C} \to \mathbf{Set}$.*

Proof. We shall construct F as the colimit in $[\mathcal{C}, \mathbf{Set}]$ of a diagram of representable functors. Explicitly, J will be a meet-semilattice[3] and $D : J \to \mathcal{C}$ a diagram such that each $D(j)$ is well-supported and each $D(j' \to j)$ is a strong epimorphism $D(j') \to D(j)$. Given such a J and diagram D, we define $F = \operatorname{colim}(J^{\mathrm{op}} \xrightarrow{D} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{Y}} [\mathcal{C}, \mathbf{Set}])$ where $\mathcal{Y}$ is the Yoneda embedding. Explicitly, elements of FA are represented by morphisms $D(j) \xrightarrow{f} A$ for some j where $f \sim f'$ if and only if the diagram

$$\begin{array}{ccc} D(j \wedge j') & \longrightarrow & D(j) \\ \downarrow & & \downarrow f \\ D(j') & \xrightarrow{f'} & A \end{array}$$

commutes.

To see that F preserves finite products, note that $F1 = \{ * \}$ and if $D(j) \xrightarrow{f} A, D(j') \xrightarrow{g} B$ represent elements of FA and FB then $D(j \wedge j') \to D(j) \xrightarrow{f} A$ and $D(j \wedge j') \to D(j) \xrightarrow{g} B$ induce an element of $F(A \times B)$ mapping to the given element of $FA \times FB$. Hence $F(A \times B) \to FA \times FB$ is surjective and it is easily seen to be injective.

We next show that F preserves equalisers. Note that if 0 exists in $\mathcal{C}$ then $F0 \neq \varnothing$. Given a parallel pair $f, g : A \rightrightarrows B$ let (E, e) be a well-supported equaliser for them. Then the equaliser of $FA \rightrightarrows FB$ consists of morphisms $D(j) \to A$ having equal composites with f and g (and hence factoring through E). If $E = 0$ then the equaliser of $FA \rightrightarrows B$ is $\varnothing$. Hence F preserves all finite limits.

Now assume that for every strong epimorphism $A \xrightarrow{f} D(j)$ in $\mathcal{C}$, there exists $j \leq j'$ such that $D(j' \to j) = f$. Then F preserves strong epimorphisms. Indeed, given a strong epimorphism $A \xrightarrow{f} B$ and a morphism $D(j) \xrightarrow{g} B$ representing an element of FB, given the pullback

$$\begin{array}{ccc} D(j') & \xrightarrow{h} & A \\ \downarrow & & \downarrow f \\ D(j) & \longrightarrow & B \end{array}$$

[3] A meet-semilattice is a partially ordered set S such that for every non-empty finite subset has a greatest lower bound. The greatest lower bound of the subset $x, y \subseteq S$ is called the **meet** of x and y and is denoted $x \wedge y$.

then h represents an element of FA whose image under Ff is g and so Ff is surjective.

Now assume that every well-supported object of $\mathcal{C}$ occurs as $D(j)$ for some j. Then F preserves properness of subobjects. Indeed, the element of FA representing $D(j) \xrightarrow{1} A$ can't be in the image of $FA' \rightarrowtail FA$ for any proper subobject $A' \rightarrowtail A$. Indeed, if it were, we would have

$$\begin{array}{ccc} D(j') & \longrightarrow & A' \\ \downarrow & \nearrow & \downarrow \\ D(j) & \xrightarrow{\;1\;} & A \end{array}$$

Since F preserves equalisers, it follows that F is faithful. Hence F reflects monomorphisms and so the argument above shows that it reflects isomorphisms.

We now construct the category J as the union $\bigcup_{n=0^\infty} J_n$ of an increasing sequence of sub-semilattices J_n. Let $J_0 = \{ 1 \}$ and $D(1) = 1$ where 1 is the terminal object of $\mathcal{C}$. Objects of $J_1 \backslash J_0$ are non-empty finite sets $A_1, \ldots, A_n$ of well-supported objects of $\mathcal{C}$ ordered by $\supseteq$ (so that $j \wedge j' = j \cup j'$) and $D(\{ A_1, \ldots, A_n) \} = \prod_{i=1}^n A_i$. Note that this is well-supported if A and B are well-supported: we have a pullback

$$\begin{array}{ccc} A \times B & \xrightarrow{\;A\;} & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & 1 \end{array}$$

If $j' \supseteq j$ then $D(j' \to j)$ is the product projection $\prod_{A \in j'} A \to \prod_{A \in j} A$. An object of $J_2 \backslash J_1$ is a pair $(j_1, \{ f_1, \ldots, f_n \})$ where $j_1 \in J_1 \backslash J_0$ and $\{ f_1, \ldots, f_n \}$ is a non-empty finite set of strong epimorphisms with codomain $D(j)$ (equivalently, well-supported objects of $\mathcal{C}/D(j_1)$. The meet of $(j_1, \{ f_1, \ldots, f_n \})$ and $(j_1', \{ g_1, \ldots, g_m \})$ has first coordinate $j_1 \wedge j_1'$ and then we take the union of the strong epimorphisms obtained by pulling back the f_i and g_i along $D(j \wedge j') \to D(j_i)$ and $D(j_i \wedge j_i') \to D(j_i')$. We define $D((j_1, \{ f_1, \ldots, f_n \}))$ to be the domain of the objet $\prod_{i=1}^n f_i$ of $\mathcal{C}/D(j_i)$ and $D(j_2 \wedge j_2') \to D(j_2)$ is the composite of the appropriate product projection in $\mathcal{C}/D(j_i \wedge j_i')$ with the appropriate pullback of $D(j_i \wedge j_i') \to D(j_i)$.

Continuing in this way, we can construct the J_n. It remains to show that we have satisfied the condition for F to preserve strong epimorphisms. If $A = D(j)$ where $j \in J_n \backslash J_{n-1}$ and $B \xrightarrow{g} A$ is a strong epimorphism then $B = D((j, \{ g \}))$ and $g = D((j, \{ g \}) \to j)$. $\square$

Remark. Let M be the monoid of endomorphisms of $F : \mathcal{C} \to \mathbf{Set}$ in $[\mathcal{C}, \mathbf{Set}]$. Then M acts on every FA so we can regard F as taking values in $[M, \mathbf{Set}]$. As such, it's still regular and faithful but is also full.

Given a regular category $\mathcal{C}$ and $S \rightarrowtail 1$ in $\mathcal{C}$, we write $\mathcal{C}_S$ for the full subcategory of $\mathcal{C}$ on objects A with $\sigma A \cong S$. This is closed in $\mathcal{C}$ under non-empty finite products, images and pullbacks of strong epimorphisms. Indeed, if we're given

$$\begin{array}{ccc} P & \xrightarrow{\;h\;} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\;g\;} & C \end{array} \quad \text{with } f \text{ strong epic then } k \text{ is strong epic so } \sigma P = \sigma B.$$

Definition 6.22. Let $\mathcal{C}$ be a category. Let $\mathcal{C}^+$ denote the category whose objects are those of $\mathcal{C}$ plus a new object 0 with one morphism $0 \to A$ and no morphisms $A \to 0$ for each $A \in \operatorname{ob} \mathcal{C}$.

Remark. In $\mathcal{C}_S^+$, every finite diagram has a limit. If it lies in $\mathcal{C}_S$ and has a limit there then that is its limit in $\mathcal{C}_S^+$. Otherwise, its limit is the unique cone with apex 0. $\mathcal{C}_S^+$ is regular. Indeed, the new morphisms $0 \to A$ are monic and hence are their own images and strong epimorphisms are still stable under pullbacks. It is also almost totally supported. Note that $\mathcal{C}/S$ may be identified with the full subcategory of $\mathcal{C}$ on objects with support $\leq S$ so its well-supported objects are those of $\mathcal{C}_S$. We have a collapsing functor $E : \mathcal{C}/S \to \mathcal{C}_S^+$ sending all objects of $\mathcal{C}_S$ to themselves and everything else to 0. E is regular and we have a regular functor $(\cdot) \times S : \mathcal{C} \to \mathcal{C}/S$ (it preserves finite limits because its right-adjoint to the forgetful functor and images because they're stable under pullback along $S \to 1$.

Theorem 6.23. *Let $\mathcal{C}$ be a small regular category. Then there exists a set I and an isomorphism-reflctin regular functor $\mathcal{C} \to \mathbf{Set}^I$.*

Proof. Let $I = \mathbf{Sub}_{\mathcal{C}}(1)$ and for each $S \in I$, consider the composite

$$G_s : \mathcal{C} \xrightarrow{(\cdot)\times S} \mathcal{C}/S \xrightarrow{E} \mathcal{C}_S^+ \xrightarrow{F_s} \mathbf{Set}$$

where F_s is the functor from the previous theorem. Then the G_s are all regular and they jointly reflect isomorphisms. Indeed, if $A \xrightarrow{f} B$ is not an isomorphism in $\mathcal{C}$, let $S = \sigma B$. Then $E(f \times S)$ is either f itself (if $\sigma(A) = S$) or $0 \to B$ (otherwise). In either case, it's not an isomorphism so its image under F_s is not an isomorphism. $\square$

7 Additive and Abelian Categories

Definition 7.1. Let $\mathcal{A}$ be a category equipped with a forgetful functor $U : \mathcal{A} \to \mathbf{Set}$. We say that a locally small category $\mathcal{C}$ is **enriched** over $\mathcal{A}$ if the hom-functor $\mathcal{C}(\cdot,\cdot) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C} \to \mathbf{Set}$ factors through U.

1. If $\mathcal{C}$ is enriched over $\mathbf{Set}_*$, we call $\mathcal{C}$ a **pointed** category.

2. If $\mathcal{C}$ is enriched over **CMon**, we call $\mathcal{C}$ a **semi-additive** category.

3. If $\mathcal{C}$ is enriched over **AbGrp**, we call $\mathcal{C}$ an **additive** category.

Moreover, functors between such categories that preserve the enrichment structures are referred to as **pointed** and **(semi-)additive**.

Remark. If $\mathcal{C}$ is a pointed category then for every pair (A, B) of objects of $\mathcal{C}$, there is a distinguished morphism $0_{A,B} \in \mathcal{C}(A, B)$ which is compatible with composition of morphisms. In other words, given morphisms $C \xrightarrow{f} A$ and $B \xrightarrow{g} C$ then $0_{A,B}f = 0_{C,B}$ and $g0_{A,B} = 0_{A,C}$.

If $\mathcal{C}$ is a semi-additive category then for every pair (A, B) of objects of $\mathcal{C}$, $\mathcal{C}(A, B)$ can be endowed with the structure of a commutative monoid and such an endowment is compatible with composition of morphisms.

If $\mathcal{C}$ is an additive category then for every pair (A, B) of objects of $\mathcal{C}$, $\mathcal{C}(A, B)$ can be endowed with the structure of an abelian group and such an endowment is compatible with composition of morphisms.

Lemma 7.2. *Let $\mathcal{C}$ be a pointed category. Given an object $I \in \mathrm{ob}\,\mathcal{C}$, the following are equivalent*

1. I is terminal

2. *I is initial*

3. $1_I = 0_{I,I}$

Proof. It suffices to show that (2) and (3) are equivalent since the equivalence of (1) and (3) is dual. To this end, suppose that I is an initial object of $\mathcal{C}$. Then there is a unique morphism $I \to I$ which is forcibly 1_I and 0_{II}.

Now suppose that $1_I = 0_{I,I}$. Fix an object $B \in \operatorname{ob}\mathcal{C}$. We need to show that there exists a unique morphism $I \to B$. Let $f : I \to B$ be a morphism. Then $f = f1_I = f0_{II} = 0_{I,B}$ and so $0_{I,B}$ is the unique morphism $I \to B$. □

Lemma 7.3. *Let $\mathcal{C}$ be a semi-additive category and $A, B, C \in \operatorname{ob}\mathcal{C}$. Then the following are equivalent*

1. *There exist morphisms $C \xrightarrow{\pi_1} A$ and $C \xrightarrow{\pi_2} B$ such that C is the product of A and B.*

2. *There exist morphisms $A \xrightarrow{\nu_1} C$ and $B \xrightarrow{\nu_2} C$ such that C is the coproduct of A and B.*

3. *There exist morphisms $A \underset{\pi_1}{\overset{\nu_1}{\rightleftarrows}} C \underset{\pi_2}{\overset{\nu_2}{\leftrightarrows}} B$ such that $\pi_1\nu_1 = 1_A, \pi_2\nu_2 = 1_B, \pi_1\nu_2 = 0_{B,A}, \pi_2\nu_1 = 0_{A,B}$ and $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$.*

Proof. It suffices to show that (1) and (3) are equivalent since the equivalence of (2) and (3) is dual. To this end, suppose that we are given morphisms $C \xrightarrow{\pi_1} A$ and $C \xrightarrow{\pi_2} B$ such that C is the product of A and B. Define η_1 and η_2 to be the unique morphisms satisfying the first four equations of the third statement. Then

$$\begin{aligned}\pi_1(\nu_1\pi_1 + \nu_2\pi_2) &= \pi_1\nu_1\pi_1 + \pi_1\nu_2\pi_2\\ &= 1_A\pi_1 + 0_{B,A}\pi_2\\ &= \pi_1 + 0_{C,A}\\ &= \pi_1\end{aligned}$$

Similarly, $\pi_2(\nu_1\pi_1 + \nu_2\pi_2) = \pi_2$. By the universal property of the product, we must then have that $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$ since it is a factorisation of the π_i through themselves.

Conversely, suppose that the third statement holds. We claim that π_1 and π_2 make C into a product of A and B. To this end, we must show that, given morphisms $D \xrightarrow{h} A$ and $D \xrightarrow{k} B$ then the cone D factors through the limit cone C uniquely. Consider the morphism $D \to C$ given by $\nu_1 h + \nu_2 k$. Then

$$\begin{aligned}\pi_1(\nu_1 h + \nu_2 k) &= \pi_1\nu_1 h + \pi_1\nu_2 k\\ &= 1_A h + 0_{B,A} k\\ &= h + 0_{D,A}\\ &= h\end{aligned}$$

Similarly, $\pi_2(\nu_1 h + \nu_2 k) = k$. It suffices to show that such a factorisation is unique. Indeed, suppose we are given another morphism $l : D \to C$ satisfying $\pi_1 l = h$ and $\pi_2 l = k$. Then

$$\begin{aligned}l = 1_C l &= (\nu_1\pi_2 + \nu_2\pi_2)l\\ &= \nu_1\pi_2 l + \nu_2\pi_2 l\\ &= \nu_1 h + \nu_2 k\end{aligned}$$

□

Definition 7.4. Let $\mathcal{C}$ be a category. We say that $0 \in \mathrm{ob}\,\mathcal{C}$ is a **zero object** if it is both initial and terminal. Moreover, given $A, B \in \mathrm{ob}\,\mathcal{C}$, we say that C is a **biproduct** of A and B if it is both a product and coproduct of A and B.

Lemma 7.5. *Let $\mathcal{C}$ be a category. If $\mathcal{C}$ has a zero object then it can be endowed with a unique pointed structure.*

Proof. Given $A, B \in \mathrm{ob}\,\mathcal{C}$, define $0_{A,B}$ to be the unique morphism $A \to 0 \to B$. We then have a unique enrichment of $\mathcal{C}$ over $\mathbf{Set}_*$ given by $(A, B) \mapsto (\mathcal{C}(A, B), 0_{A,B})$. □

Lemma 7.6. *Let $\mathcal{C}$ be a pointed category with all finite products and coproducts. If for every pair of objects (A, B) the canonical morphism*

$$c : A + B \to A \times B$$

defined by $\pi_i c \nu_j = \delta_{ij}$ is an isomorphism then $\mathcal{C}$ admits a unique semi-additive structure.

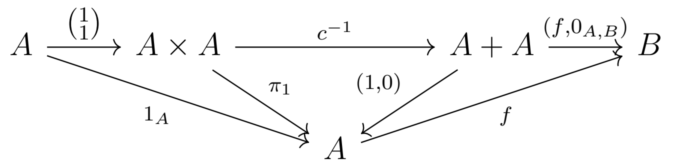
Proof. Given a parallel pair $f, g : A \rightrightarrows B$ let $f +_L g$ denote the composite

$$A \xrightarrow{\binom{1}{1}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{(f,g)} B$$

and $f +_R g$ the composite

$$A \xrightarrow{\binom{f}{g}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{(1,1)} B$$

We claim that $+_L = +_R$ and is the unique semi-additive structure on $\mathcal{C}$. It follows immediately by construction that $h(f +_L g) = hf +_L hg$ and $(f +_R g)k = fk +_R gk$ whenever such compositions are defined. We next show that $f +_L 0_{A,B} = f$. Indeed, consider the diagram



Now, all three triangles in this diagram are clearly commutative whence $f +_L 0_{A,B} = f$. Similarly, $0_{A,B} +_L f$ and dually for $+_R$. Now consider the composite

$$A \xrightarrow{\binom{1}{1}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{\left(\begin{smallmatrix} f & g \\ h & k \end{smallmatrix}\right)} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{(1,1)} B$$

The composite $A \to B \times B$ is $\binom{f +_L g}{h +_L k}$ so that the above is equal to $(f +_R h) +_L (g +_R h)$. On the other hand, it also equals $(f +_R h) +_L (g +_R h)$.

Setting $g = h = 0$ yields $f +_R h = f +_L h$ so that $+_R = +_L$. Moreover, setting $f = k = 0$ yields $g + h = h + g$ so $+ = +_L = +_R$ is commutative. Setting $g = 0$ yields $f + (h + k) = (f + h) + k$ so $+$ is associative. Hence $+$ defines a semi-additive structure on $\mathcal{C}$.

To show that it is the unique semi-additive structre on $\mathcal{C}$, let $+_a$ be another semi-additive structure on $\mathcal{C}$. By Lemma 7.3 we have

$$c^{-1} = \nu_1 \pi_1 +_a \nu_2 \pi_2$$

so that the definitions of $+_L$ and $+_R$ coincide with $+_a$. □

Corollary 7.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be semi-additive categories with finite biproducts. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is semi-additive if and only if it preserves finite biproducts.*

Proof. First suppose that F is semi-additive. Let $C \xrightarrow{\pi_1} A$ and $C \xrightarrow{\pi_2} B$ be such that C is a product of A and B. By Lemma 7.3, there exist morphisms ν_1 and ν_2 fitting in the diagram $A \underset{\pi_1}{\overset{\nu_1}{\rightleftarrows}} C \underset{\pi_2}{\overset{\nu_2}{\leftrightarrows}} B$ such that $\pi_1\nu_1 = 1_A, \pi_2\nu_2 = 1_B, \pi_1\nu_2 = 0, \pi_2\nu_1 = 0$ and $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$. Since F is semi-additive, we may apply F on these definitions and relations to yield morphisms satisfying the same conditions in $\mathcal{D}$. We may then conclude by the equivalence of Lemma 7.3 that $F\pi_1$ and $F\pi_2$ make FC a product of FA and FB so that F preserves finite products. The fact that F preserves finite coproducts follows by a dual argument. Hence F preserves finite biproducts.

Now suppose that F preserves finite biproducts. We need to show that $F(0) = 0$ and $F(f+g) = Ff + Fg$. Since the zero object can be realised as an empty biproduct, it follows immediately that $F(0) = 0$. By Lemma 7.6, we may apply F across the diagram defining $+$ to get

$$FA \xrightarrow{\binom{1}{1}} FA \times FA \xrightarrow{Fc^{-1}} FA + FA \xrightarrow{(Ff,Fg)} FB$$

which is exactly $Ff + Fg$ as required. □

Definition 7.8 ($\star$)**.** Let $\mathcal{C}$ be a pointed category and $A \xrightarrow{f} B$ a morphism in $\mathcal{C}$. We define the **kernel** of f, denoted $\ker f$, to be the equaliser of f and 0. Moreover, we say that a monomorphism in $\mathcal{C}$ is **normal** if it occurs as a kernel.

Proposition 7.9 ($\star$)**.** *Let $\mathcal{C}$ be an additive category. Then every regular monomorphism is normal.*

Proof. Let $e : A \to B$ be a regular monomorphism. Recall that this means that e occurs as the equaliser of two morphisms f and g, say. By definition of the equaliser, we have that $fe = ge$ and e is universal amongst such morphisms. By the additive structure of $\mathcal{C}$ we then have that $fe - ge = 0$ whence $(f - g)e = 0 = 0e$. Hence e is the kernel of $f - g$ and 0 whence e is normal. □

Definition 7.10. Let $\mathcal{C}$ be an additive category and $A \xrightarrow{f} B$ a morphism in $\mathcal{C}$. We say that f is a **pseudo-monomorphism** if $\ker f$ occurs as a zero map. In other words, we have $fg = 0$ implies that $g = 0$.

Example 7.11. In **Grp** every epimorphism $f : G \to H$ is normal since it occurs as the cokernel of $\ker f \hookrightarrow G$. However, not every monomorphism is normal.

Example 7.12. In **Set**$_*$ ever monomorphism is normal. Indeed, if $(A, a) \xrightarrow{f} (B, b)$ is a monomorphism then it occurs as the kernel of $(B, b) \to (C, b)$ where $C = (B \setminus \operatorname{im} f) \cup \{ b \}$. However, not every epimorphism is normal. A normal epimorphism is bijective on elements not mapped to the basepoints.

Lemma 7.13 ($\star$)**.** *Let $\mathcal{C}$ be a pointed category with kernels and cokernels. Then a morphism f is normal if and only if $f = \ker \operatorname{coker} f$.*

Proof. The backwards direction is trivial. Hence suppose that $f : A \to B$ is normal so that $f = \ker g$ for some morphism $g : B \to C$. Then $gf = 0$ so that g coequalises f so it must factor as $h \operatorname{coker} f$. Now suppose we have a morphism $k : A \to B$ equalising $\operatorname{coker} f$ and 0 so that $\operatorname{coker}(f)k = 0$. Then $gk = h \operatorname{coker}(f)k = h0 = 0$. Hence k must factor uniquely through $f = \ker g$, say $k = fl$. But k was an arbitrary morphism equalising $\operatorname{coker} f$ and 0 so that $f = \ker \operatorname{coker} f$. □

Lemma 7.14 ($\star$)**.** *Let $\mathcal{C}$ be a pointed category with kernels and cokernels and such that every monomorphism is normal. Then $\mathcal{C}$ has images. In particular, the image of a morphism f is exactly* $\ker \operatorname{coker} f$.

Proof. Fix a morphism $A \xrightarrow{f} B$. Recall that f has an image if there exists a minimal subobject of B through which f factors. Now note that f factors through $\ker(B \xrightarrow{g} C)$ if and ony if $gf = 0$ if and only if g factors through $\operatorname{coker} f$ if and only if $\ker \operatorname{coker} f$ factors through $\ker g$ so that $\ker \operatorname{coker} f$ is the smallest subobject of B through which f factors. □

Definition 7.15. Let $\mathcal{A}$ be a category. We say that $\mathcal{A}$ is **abelian** if

1. $\mathcal{A}$ is additive.
2. $\mathcal{A}$ has finite biproducts, kernels and cokernels.
3. All monomorphisms and epimorphisms of $\mathcal{A}$ are normal.

Example 7.16. AbGrp is abelian. More generally, so is $\mathbf{Mod_R}$ for any commutative ring R.

Example 7.17. If $\mathcal{A}$ is abelian and $\mathcal{C}$ is an arbitrary category then $[\mathcal{C}, \mathcal{A}]$ is abelian with everything defined pointwise.

Example 7.18. Let $\mathcal{A}$ be abelian and $\mathcal{C}$ be additive. Then the full subcategory $\mathbf{Add}(\mathcal{C}, \mathcal{A})$ of $[\mathcal{C}, \mathcal{A}]$ consisting of additive functors $\mathcal{A} \to \mathcal{A}$ is abelian. We note that $\mathbf{Mod_R} = \mathbf{Add}(R, \mathbf{AbGrp})$ where we regard R as an additive category with one object.

Example 7.19. tfAbGrp is additive but not abelian. Indeed, it does not possess cokernels.

Lemma 7.20. *Let $\mathcal{C}$ be an additive category with finite biproducts and consider the (not necessarily commutative) square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{k} & D \end{array}$$

1. *The above square is commutative if and only if the* ***flattening*** *composition*

$$A \xrightarrow{\binom{f}{g}} B + C \xrightarrow{(h,-k)} D$$

is zero.

2. *The above square is a pullback if and only if $\binom{f}{g} = \ker(h, -k)$.*
3. *The above square is a pushout if and only if $(h, -k) = \operatorname{coker} \binom{f}{g}$*

Proof. The flattening is simply the morphism $fh - gk$ which is 0 if and only if $fh = gk$.

To see thet second part, note that $\ker(h, -k)$ is a morphism $\binom{f}{g}$ such that $(h, -k)\binom{f}{g} = fh - gk = 0$ and universal among such. But this is exactly what it means for the above diagram to be a pullback.

The third part follows dually. □

Corollary 7.21. *Let $\mathcal{A}$ be an abelian category. Then epimorphisms in $\mathcal{A}$ are stable under pullback.*

Proof. Suppose we are given a pullback square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{k} & D \end{array}$$

with h epic. We first claim that this square is also a pushout. By Lemma 7.20, it suffices to show that $(h, -k) = \operatorname{coker}\binom{f}{g}$. By the same Lemma, we have that $\binom{f}{g} = \ker(h, -k)$. But $(h, -k)$ is epic since h is. Since $\mathcal{A}$ is abelian, $(h, -k)$ is then normal epic whence by Lemma 7.13 we see that $(h, -k) = \operatorname{coker}\binom{f}{g}$ as claimed.

We now claim that g is epic. Since $\mathcal{A}$ is abelian, it suffices to show that g is pseudo-epic. Suppose we have a morphism $C \xrightarrow{l} E$ such that $lg = 0$. Then the pair $(l, B \xrightarrow{0} E)$ form a cone under (f, g) so they must factor through (k, h), say by $m : D \to E$. Then $mh = 0$ whence $m = 0$ since h is epic. Hence $l = mk = 0$ whence g is epic. □

Theorem 7.22. *Let $\mathcal{A}$ be a category. Then $\mathcal{A}$ is abelian if and only if it is effective regular.*

Proof. First suppose that $\mathcal{A}$ is abelian. By Corollary 7.21, $\mathcal{A}$ is regular so we just need to show that it is effective. By definition, we need to show that every equivalence relation $(a, b) : R \rightrightarrows A$ occurs as the kernel pair of some morphism in $\mathcal{C}$. Consider the pullback diagram

$$\begin{array}{ccc} K & \overset{l}{\rightarrowtail} & R \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle (a,b)} \\ A & \xrightarrow{\binom{1}{0}} & A + A \end{array}$$

where we have used Lemma 7.20 to see that k and l are monid. Since k is monic and $\mathcal{A}$ is abelian, it is the kernel of some $A \xrightarrow{f} B$. We claim that (a, b) is the kernel pair of f. Suppose we are given a parallel pair $x, y : C \rightrightarrows$ such that $fx = fy$. We need to show that (x, y) factors through (a, b). Observe that $x - y$ equalises f and 0 so that $x - y$ factors uniquely as kz for some $x : C \to K$. Since (a, b) is an equivalence relation, we may fix an $r : A \to R$ such that $ar = br = 1_A$. Consider $lz + ry : C \to R$. We have

$$a(lz + ry) = alz + ary = kz + y = x - y + y = xb(lz + ry) \quad = blz + bry = 0z + y = y$$

so that $(lz + ry)$ is a factorisation of (x, y) through (a, b).

Conversely, we shall first show that any reflexive relation in an additive category with finite limits is symmetric and transitive. To this end, suppose that we are given a relation $(a, b) : R \rightrightarrows A$ and a morphism $r : A \to R$ such that $ar = br = 1_A$. To see that (a, b) is symmetric, consider the morphism $s = ra + rb - 1_R : R \to R$. Then

$$\begin{aligned} as &= ara + arb - a = a + b - a = b \\ ba &= bra + brb - b = a + b - b = a \end{aligned}$$

To see that (a, b) is transitive, consider the pullback square

$$\begin{array}{ccc} T & \xrightarrow{q} & R \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle a} \\ R & \xrightarrow{b} & A \end{array}$$

Let $t = p + q - raq : T \to R$. Then

$$at = ap + aq - araq = ap + aq - aq = ap$$
$$bt = bp + bq - brbp = bp + bq - bp = bq$$

Now suppose that $\mathcal{A}$ is additive and effective regular. Since $\mathcal{A}$ is regular, it has all finite limits and hence kernels. Moreover, Lemma 7.3 implies that $\mathcal{A}$ then has all finite biproducts.

We next show that $\mathcal{A}$ has cokernels. Since $\mathcal{A}$ has images, it suffices to show that all monomorphisms have cokernels. To this end, fix a monomorphism $K \rightarrowtail kA$ and consider the diagram

$$K + A \underset{(0,1)}{\overset{(k,1)}{\rightrightarrows}} A$$

This is jointly monic since if $(k,1)\binom{x}{y} = (0,1)\binom{x}{y} = 0$ then $y = 0$ and $kx = 0$ so $x = 0$. Moreover its reflexive with common splitting $A \xrightarrow{\binom{0}{1}} K + A$. By the discussion above, we have that $(k,1)$ and $(0,1)$ form an equivalence relation. Since $\mathcal{A}$ is effective, $(k,1)$ and $(0,1)$ occur as the kernel pair of some morphism. Since coequalisers in regular categories are kernel pairs, it follows that this pair of morphisms has a coequaliser. But such a coequaliser is a cokernel for κ.

We now show that every monomorphism is normal. Consider the same diagram as above leading to a kernel pair. Now suppose that $fg = 0$ for some morphism $C \xrightarrow{g} A$. Then $(g,0)$ factors as

$$C \xrightarrow{\binom{u}{v}} K + A \underset{(0,1)}{\overset{(k,1)}{\rightrightarrows}} A$$

by the universal property of the coproduct. Hence $v = 0$ whence $ku = g$ and so $k = \ker f$.

Finally, we must show that every epimorphism is normal. Let $A \xrightarrow{f} B$ be an epimorphism. Then we can write $A \xrightarrow{q} I \xrightarrow{m} B$ where I is the image of f, q is a normal epimorphism and m is both regular monic and epic. m is then an isomorphism whence f is normal. $\square$

Definition 7.23. Let $\mathcal{C}$ be a pointed category with kernels and cokernels. Consider a sequence

$$\dots \longrightarrow C_n \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow \dots$$

of objects and morphisms in $\mathcal{C}$. We say that such a sequence is **exact** at C_n if $\operatorname{im} f_{n+1} = \ker f_n$ or, equivalently, if $\operatorname{coker} f_{n+1} = \operatorname{coim} f_n$.

Example 7.24. Let $\mathcal{A}$ be an abelian category. Then $0 \to A \xrightarrow{f} B$ is exact if and only if f is monic.

Now consider the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$. By the above, this is exact at A if and only if f is a monomorphism. It then follows that the sequence is exact at B if and only if $\ker g = f$. Indeed, in an abelian category, evey monomorphism is normal. By Lemma 7.13 we have that f is a normal monomorphism if and only if $f = \ker \operatorname{coker} f$. Appealing to Lemma 7.14 we then see that $\operatorname{im} f = f$ as claimed.

Now consider the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. By the above, we know that this sequence is exact at A if and only if f is monic and exact at B if and only if $f = \ker g$. By an argument dual to the above, we see that the sequence $B \xrightarrow{g} C \to 0$ is exact at C if and only if g is an epimorphism. It then follows that the original sequence is exact at C if and only if

$g = \operatorname{coker} f$. Indeed, in an abelian category, every epimorphism is normal. By Lemma 7.13 we have that g is a normal epimorphism if and only if $g = \operatorname{coker} \ker g$. Appealing to Lemma 7.14 we see that $\operatorname{coim} g = g$ whence $\operatorname{coker} f = g$.

Definition 7.25. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. We say that a functor $F : \mathcal{A} \to \mathcal{B}$ is **left exact** if it preserves kernels and **right exact** if it preserves cokernels. Moreover, we say that F is **exact** if it is both left and right exact so that it preserves exact sequences.

Remark. Note that a biproduct $A + B$ is characterised by the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{\binom{1}{0}} A + B \xrightarrow{(0,1)} B \longrightarrow 0$$

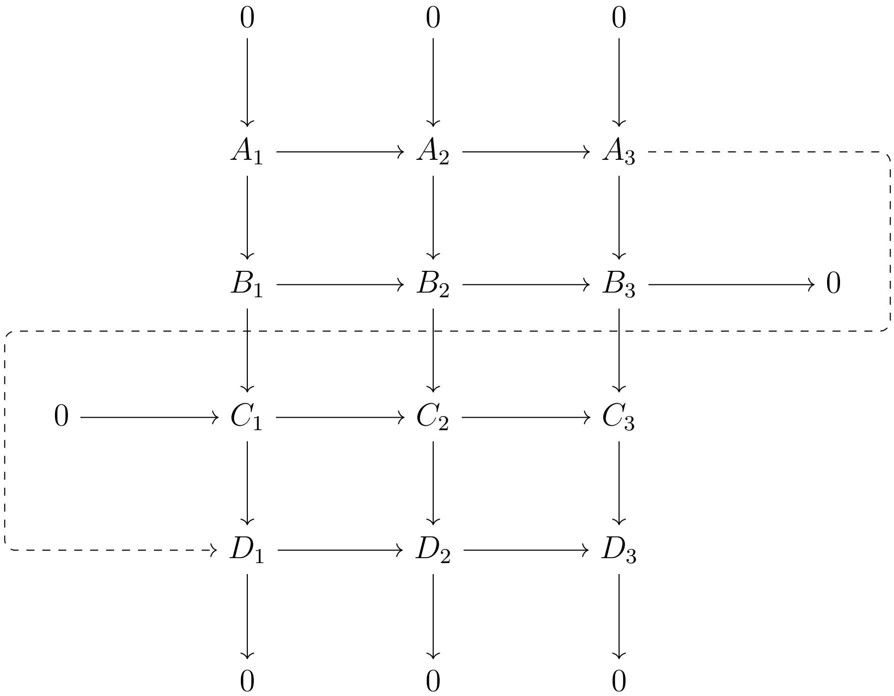
together with the fact that $\binom{1}{0}$ and $(0, 1)$ are split. Hence any exact functor preserves biproducts.

We further remark that if F is left exact and preserves epimorphisms then it is exact. In particular, a regular functor between abelian categories is exact.

Theorem 7.26. *Let $\mathcal{A}$ be a small abelian category. Then there exists a faithful exact functor $\mathcal{A} \to$ **AbGrp**. Moreover, there exists a fully faithful exact functor $\mathcal{A} \to \mathbf{Mod_R}$ for some ring R.*

Proof. We first observe that $\mathcal{A}$ is totally supported. Indeed, the support of an object A is clearly isomorphic to the zero object of $\mathcal{A}$. By Theorem 6.21, there exists an isomorphism reflecting functor $F : \mathcal{C} \to \mathbf{Set}$. Recall that we defined $F = \operatorname{colim}_J \mathcal{A}(D(j), \cdot) : \mathcal{A} \to \mathbf{Set}$. Since $\mathcal{A}$ is an abelian category it is enriched over **AbGrp** so we may view each $\mathcal{A}(D(j), \cdot)$ as functors $\mathcal{A} \to \mathbf{AbGrp}$. We may thus view view their colimit as living in $[\mathcal{A}, \mathbf{AbGrp}]$. The description of F from Theorem 6.21 still works. Indeed, it is still regular since $\mathbf{AbGrp} \to \mathbf{Set}$ reflects finite limits and strong epimorphisms so it is exact. Moreover, it is still faithful. To obtain the full and faithful functor to $\mathbf{Mod_R}$, we may repeat the process in Remark 6. □

Lemma 7.27 (Snake Lemma)**.** *Let $\mathcal{A}$ be an abelian category and suppose we are given a commutative diagram*



with exact rows and columns. Then there exists a dotted morphism as above forming an exact sequence.

Proof. Proof ommitted. ☐

Definition 7.28 ($\star$)**.** Let $\mathcal{A}$ be an abelian category. We define a **chain complex** in $\mathcal{A}$ to be a sequence

$$\dots \longrightarrow C_n \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \dots$$

such that $\delta_n \delta_{n+1} = 0$ for all n. The complexes in $\mathcal{A}$ form an abelian category $c\mathcal{A}$ whose objects are the complexes $C_\bullet$ in $\mathcal{A}$ and whose morphisms consist of a collection of morphisms $\{ f_n : C_n \to D_n \}_{n \in \mathbb{Z}}$ such that the diagram

$$\begin{array}{ccc} C_{n-1} & \xrightarrow{\delta_n} & C_n \\ \big\downarrow{\scriptstyle f_{n-1}} & & \big\downarrow{\scriptstyle f_n} \\ D_{n-1} & \xrightarrow{\delta_{n-1}} & D_n \end{array}$$

commutes for all n.

Remark. Note that we actually have $c\mathcal{A} = \mathbf{Add}(\mathcal{Z}, \mathcal{A})$ where $\mathcal{Z}$ is the additive category with $\operatorname{ob} \mathcal{Z} = \mathbb{Z}$, morphisms given by $\mathcal{Z}(p, q) = \mathbb{Z}$ if $q = p$ or $p - 1$ and $\{ 0 \}$ otherwise and composition given by $p \xrightarrow{m} q \xrightarrow{n} r = mn$ if $p \geq q \geq r \geq p - 1$ or 0 otherwise.

Definition 7.29. Let $\mathcal{A}$ be an abelian category and $C_\bullet$ a complex in $\mathcal{A}$. We define

1. the $\mathbf{n^{th}}$**-cycles** object $Z_n(C_\bullet) = \ker \delta_n$.
2. the $\mathbf{n^{th}}$**-boundaries** object $B_n(C_\bullet) = \operatorname{im} \delta_{n+1}$.
3. the $\mathbf{n^{th}}$**-homology** object $H_n(C_\bullet) = \operatorname{coker}(B_n \rightarrowtail Z_n)$.

If we denote $Q_n(C_\bullet) = \operatorname{coker} f_{n+1}$ then these definitions fit into the diagram

$$\begin{array}{ccccccccccccc} C_{n+1} & & & \xrightarrow{\delta_{n+1}} & & & C_n & & & \xrightarrow{\delta_n} & & & C_{n-1} \\ & \searrow & & & & \nearrow & & \searrow & & & & \nearrow & \\ & & Q_{n+1} & \twoheadrightarrow & B_n & \rightarrowtail & Z_n & \twoheadrightarrow & H_n & \rightarrowtail & Q_n & \twoheadrightarrow & B_{n-1} \rightarrowtail Z_{n-1} \end{array}$$

Theorem 7.30 (Mayer-Vietoris)**.** *Let $\mathcal{A}$ be an abelian category and*

$$0 \longrightarrow C_\bullet \longrightarrow D_\bullet \longrightarrow E_\bullet \longrightarrow 0$$

a short exact sequence of complexes in $c\mathcal{A}$. Then there exists a long exact sequence of homology objects in $\mathcal{A}$

$$
\begin{array}{l}
H_n(C_\bullet) \longrightarrow H_n(D_\bullet) \longrightarrow H_n(E_\bullet) \dashrightarrow \\
\dashrightarrow H_2(C_\bullet) \longrightarrow H_2(D_\bullet) \longrightarrow H_2(E_\bullet) \longrightarrow \\
\longrightarrow H_1(C_\bullet) \longrightarrow H_1(D_\bullet) \longrightarrow H_1(E_\bullet) \longrightarrow \\
\longrightarrow H_0(C_\bullet) \longrightarrow H_0(D_\bullet) \longrightarrow H_0(E_\bullet) \longrightarrow 0
\end{array}
$$

Proof. We first apply the Snake Lemma to the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Z_n(C_\bullet) & \longrightarrow & Z_n(D_\bullet) & \longrightarrow & Z_n(E_\bullet) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n & \longrightarrow & D_n & \longrightarrow & E_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1} & \longrightarrow & D_{n-1} & \longrightarrow & E_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Q_{n-1}(C_\bullet) & \longrightarrow & Q_{n-1}(D_\bullet) & \longrightarrow & Q_{n-1}(E_\bullet) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

To obtain an exact sequence

$$
\begin{array}{l}
0 \longrightarrow Z_n(C_\bullet) \longrightarrow Z_n(D_\bullet) \longrightarrow Z_n(E_\bullet) \longrightarrow \\
\longrightarrow Q_{n-1}(C_\bullet) \longrightarrow Q_{n-1}(D_\bullet) \longrightarrow Q_{n-1}(E_\bullet) \longrightarrow 0
\end{array}
$$

This gives us another diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_n(C_\bullet) & \longrightarrow & H_n(D_\bullet) & \longrightarrow & H_n(E_\bullet) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Q_n & \longrightarrow & Q_n & \longrightarrow & Q_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & Z_{n-1} & \longrightarrow & Z_{n-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_{n-1}(C_\bullet) & \longrightarrow & H_{n-1}(D_\bullet) & \longrightarrow & H_{n-1}(E_\bullet) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Applying the Snake Lemma to this diagram then yields the desired sequence. □

Definition 7.31. Let $\mathcal{A}$ be an abelian category and $f_\bullet, g_\bullet : C_\bullet \rightrightarrows D_\bullet$ morphisms of complexes. By a **homotopy** of $f_\bullet$ and $g_\bullet$, we mean a sequence of morphisms $h_n : C_n \to D_{n+1}$ such that $f_n - g_n = d_{n+1}h_n + h_{n-1}c_n$ for all n. If there exists a homotopy of $f_\bullet$ and $g_\bullet$ then we write $f_\bullet \simeq g_\bullet$.

Remark. We remark that a homotopy is an equivalence relation and so we can form the quotient category $c\mathcal{A}/\simeq$.

Lemma 7.32 ($\star$)**.** *Let $\mathcal{A}$ be an abelian category and $f_\bullet \simeq g_\bullet$ a homotopy of morphisms of complexes. Then $H_n(f_\bullet) = H_n(g_\bullet)$.*

Proof. Observe that the the difference $Z_n(f_\bullet) - Z_n(g_\bullet)$ is the restriction to $Z_n(C_\bullet)$ of $d_{n+1}h_n + h_{n-1}c_n$. But the second term vanishes since the restriction of c_n to $Z_n(C_\bullet)$ is just 0. Composing with the quotient map $Z_n(D_\bullet) \to H_n(D_\bullet)$ kills the first term $d_{n+1}h_n$ and so $H_n(f_\bullet) - H_n(g_\bullet) = 0$. □

Definition 7.33 ($\star$)**.** Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ has **enough projectives** if for all objects $A \in \operatorname{ob}\mathcal{C}$ there exists an epimorphism $P \twoheadrightarrow A$ with P projective.

Definition 7.34 ($\star$)**.** Let $\mathcal{A}$ be an abelian category and $A \in \operatorname{ob}\mathcal{C}$ an object. We say that a sequence

$$\ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

in $\mathcal{A}$ is a **projective resolution** if each P_i is projective.

Lemma 7.35 ($\star$)**.** *Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object of $\mathcal{A}$ has a projective resolution.*

Proof. Fix an object $A \in \operatorname{ob}\mathcal{A}$ and choose a $P_0 \twoheadrightarrow A$ with P_0 projective. Let $K_0 \to P_0 = \ker(P_0 \to A)$ and choose $P_1 \twoheadrightarrow K_0$ with P_1 projective. Continuing in this fashion, we can construct a projective resolution of A. □

Lemma 7.36 ($\star$)**.** *Let $\mathcal{A}$ be an abelian category, $A, B \in \operatorname{ob}\mathcal{A}$ and $P_\bullet, Q_\bullet$ projective resolutions of A and B respectively. Then for all $f : A \to B$ there exists $f_\bullet : P_\bullet \to Q_\bullet$ such that $H_0(f_\bullet) = f$ and any such morphisms of chain complexes are unique up to homotopy.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & A \\
 & & & & & & \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f} \\
\ldots & \longrightarrow & Q_2 & \xrightarrow{q_2} & Q_1 & \xrightarrow{q_1} & Q_0 & \xrightarrow{q_0} & B
\end{array}$$

Since P_0 is projective, there exists a morphism f_0 making the right hand square of the diagram commute. Now, $0 = fp_0p_1 = q_0f_0p_1$ so that f_0p_1 factors through $\ker q_0 = \operatorname{im} q_1$, say by $k : P_1 \to \operatorname{im} q_1$. We thus have the following diagram

$$\begin{array}{ccccc}
P_1 & & & & \\
\Big\downarrow{\scriptstyle f_1} & \searrow{\scriptstyle k} & & \searrow{\scriptstyle p_1 f_0} & \\
Q_1 & \twoheadrightarrow & \operatorname{im} q_1 & \rightarrowtail & Q_0
\end{array}$$

Since P_1 is projective, we therefore have a morphism $f_1 : P_1 \to Q_1$ making this diagram commute. Continuing in this fashion, we can construct a morphism of chain complexes $f_\bullet : P_\bullet \to Q_\bullet$. It then follows immediately that $H_0(f_\bullet) = f$.

Now let $g_\bullet : P_\bullet \to Q_\bullet$ be another morphism of projective resolutions. Consider the diagram

$$\begin{array}{ccc} P_0 & \xrightarrow{p_0} & A \\ {\scriptstyle g_0}\downarrow\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ Q_0 & \xrightarrow{q_0} & B \end{array}$$

Then $q_0(f_0 - g_0) = fp_0 - fp_0 = 0$ so that $f_0 - g_0$ factors through $\ker q_0 = \operatorname{im} q_1$, say by k. We then have the diagram

$$\begin{array}{ccccc} P_0 & & & & \\ \vdots{\scriptstyle h_0} & \searrow^{k} & & \searrow^{f_0-g_0} & \\ Q_1 & \twoheadrightarrow & \operatorname{im} q_1 & \rightarrowtail & Q_0 \end{array}$$

Since P_1 is projective, there exists a morphism $h_0 : P_0 \to Q_1$ making this diagram commute. In other words, we have $f_0 - g_0 = q_1 h_0$. Now,

$$\begin{aligned} q_1(f_1 - g_1 - h_0 p_1) &= q_1 f_1 - q_1 g_1 - q_1 h_0 p_1 \\ &= f_0 p_1 - g_0 p_1 - (f_0 - g_0) p_1 \\ &= f_0 p_1 - g_0 p_1 - f_0 p_1 + g_0 p_1 \\ &= 0 \end{aligned}$$

so that $f_1 - g_1 - h_0 p_1$ factors through $\ker q_1 = \operatorname{im} q_2$, say by k. We then have the diagram

$$\begin{array}{ccccc} P_1 & & & & \\ \vdots{\scriptstyle h_1} & \searrow^{k} & & \searrow^{f_1-g_1-h_0p_1} & \\ Q_2 & \twoheadrightarrow & \operatorname{im} q_2 & \rightarrowtail & Q_1 \end{array}$$

Since P_1 is projective, there exists a morphism $h_1 : P_1 \to Q_2$ making the above diagram commute. In other words, $q_2 h_1 = f_1 - g_1 - h_0 p_1$ and so $f_1 - g_1 = q_2 h_1 + h_0 p_1$. Continuing in this fashion, we may construct a homotopy $h_\bullet : f_\bullet \to g_\bullet$. □

Remark. From this Lemma we see that projective resolutions of objects in an abelian category are unique up to homotopy.

Definition 7.37. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives. Given an additive functor $F : \mathcal{A} \to \mathcal{B}$ we define the **left-derived functors** of F to be $L^\bullet F(A) = H_\bullet(FP(A))$ where $P(A)$ is a projective resolution of $A \in \operatorname{ob} \mathcal{A}$.

Remark. This is well-defined since projective resolutions are unique up to homotopy and homology objects are invariant under homotopy.

Moreover, if F is an exact functor then $L^0 F \cong F$ and $L^n F = 0$ for all $n > 0$. If F is right exact then we still have that $L^0 F \cong F$ since $FP_1 \to FP_0 \to FA \to 0$ is exact but it may be the case that $L^n F$ are zero for $n > 0$.

Theorem 7.38. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories such that $\mathcal{A}$ has enough projectives. If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor and*

$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} 0$$

is a short exact sequence in $\mathcal{A}$ then there exists a long exact sequence of left-derived functors

$$\begin{array}{ccccccc}
L^nFA & \longrightarrow & L^nFB & \longrightarrow & L_nFC & \dashrightarrow & \\
 & & \vdots & & & & \\
L^2FA & \longrightarrow & L^2FB & \longrightarrow & L^2FC & \to & \\
L^1FA & \longrightarrow & L^1FB & \longrightarrow & L^1FC & \to & \\
L^0FA & \longrightarrow & L^0FB & \longrightarrow & L^0FC & \longrightarrow & 0
\end{array}$$

Proof. We first claim that, given projective resolutions $P_\bullet$ of A and $R_\bullet$ of C, there exists a projective resolution $Q_\bullet$ of B such that $Q_n = P_n + R_n$ and

$$P_n \xrightarrow{\binom{1}{0}} P_n + R_n \xrightarrow{(0,1)} R_n$$

are morphisms of chain complexes. We first observe that we have the following diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & A \\
 & & \downarrow{\scriptstyle\binom{1}{0}} & & \downarrow{\scriptstyle\binom{1}{0}} & & \downarrow{\scriptstyle\binom{1}{0}} & & \downarrow{\scriptstyle f} \\
 & & P_2 + R_2 & & P_1 + R_1 & & P_0 + R_0 & \xrightarrow{(fp_0,t)} & B \\
 & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle (0,1)} & \nearrow{\scriptstyle t} & \downarrow{\scriptstyle g} \\
\cdots & \longrightarrow & R_2 & \xrightarrow{r_2} & R_1 & \xrightarrow{r_1} & R_0 & \xrightarrow{r_0} & C
\end{array}$$

Since R_0 is projective, there exists a morphism $t : R_0 \to B$ so that $r_0 = gt$. We claim that (fp_0, t) is an epimorphism. Since $\mathcal{B}$ is abelian, it suffices to show that it is pseudo-epic. To this end, suppose that $x(fp_0, t) = 0$. Then $xfp_0 = 0$. Since p_0 is epic, it follows that $xf = 0$. Hence x factors through $\operatorname{coker} f = g$, say by y. Then $0 = xt = xygt = yr_0$. But r_0 is epic whence $y = 0$ and so $x = 0$.

Now let K_0, L_0, M_0 be the kernels of $p_0, (fp_0, t)$ and r_0 respectively. By the snake lemma, we have that $0 \to K_0 \to L_0 \to M_0 \to 0$ is exact. Then we have the following diagram

$$\begin{array}{ccccccc}
P_1 & \twoheadrightarrow & K_0 & \rightarrowtail & P_0 & \xrightarrow{p_0} & A \\
\downarrow{\scriptstyle\binom{1}{0}} & & \downarrow & & \downarrow{\scriptstyle\binom{1}{0}} & & \downarrow{\scriptstyle f} \\
P_1 + R_1 & \dashrightarrow & L_0 & \rightarrowtail & P_0 + R_0 & \xrightarrow{(fp_0,t)} & B \\
\downarrow{\scriptstyle (0,1)} & & \downarrow & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle g} \\
R_1 & \twoheadrightarrow & M_0 & \rightarrowtail & R_0 & \xrightarrow{r_0} & C
\end{array}$$

In exactly the same fashion as before, we can construct an epimorphism $P_1 + R_1 \to L_0$. Continuing in this way, we construct the claimed projective resolution of B.

Now, F is an additive functor and so preserves the exactness of the columns $0 \to P_n \to Q_n \to R_n \to 0$. The result then follows from the Mayer-Vietoris Theorem applied to the exact sequence $0 \to FP_\bullet \to FQ_\bullet \to FR_\bullet \to 0$. $\square$